

# Chapter 11

## Consumption

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### 11.1 Introduction

Household consumption is a key determinant of welfare and, as a result, it plays a fundamental role in many areas of macroeconomics, such as growth, business cycles, inequality, taxation, and asset pricing. The growth of per-capita consumption over time is an unequivocal sign of a society's rising prosperity. Because consumption fluctuations over the business cycle are costly to households, both fiscal and monetary policy go to great lengths in order to stabilize aggregate consumption expenditures. The distribution of consumption in the population is a credible measure of inequality in standard of living across households, more so than income. Crucial objectives of redistributive and social insurance policies are that of supporting a minimum level of consumption above poverty for all households, and that of limiting the pass through of income losses to household spending (and thus, to their well-being). Finally, consumption choices of investors over time and across states determine stochastic discount factors which price assets in financial markets.

In light of this centrality, it is not surprising that the theory and empirics of consumption choices has, historically, attracted so much attention from economists. As [Deaton \(1992\)](#) puts it, in the preface of his book, *attempts by economists to understand the saving and consumption patterns of households have generated some of the best science in economics*. The desire to microfound the empirical relation between consumption and income, which contradicted the simple Keynesian consumption function where expenditures are modelled as a linear function of current income, sparked some of the first examples of forward-looking dynamic optimization ([Modigliani and Brumberg, 1954](#); [Friedman, 1957](#)) and, since then, led to gradual enrichment of the optimization framework. At the same time, the ever wider availability of large microeconomic data sets on income and expenditures (first survey data, now “big” administrative and proprietary data) created fertile grounds for the application of state-of-the-art econometric techniques to test models and quantify key magnitudes. In the last 30 years, these theoretical and empirical advances have been incorporated into general equilibrium models with heterogeneous households that constitute one of the main workhorses for the study of business cycle, inequality, and government policy in macroeconomics.

This chapter offers an introduction to this vast literature. It emphasizes the theoretical advancements in this field, but it also makes an attempt to relate models to data. The chap-

ter is organized as follows. Section 11.2 derives consumption allocations under two extreme financial market structures: autarky and complete markets. It then argues that the empirical evidence suggests a market structure in between these two which offers “partial insurance” against income shocks, and introduces an economy where only a non-state-contingent bond is traded. Section 11.3 studies in some depth the optimal intertemporal consumption/saving problem of a household who can save and borrow through this asset, the so called “income fluctuation problem.” We start from the deterministic case, and then we analyze the stochastic case with random income fluctuations. The permanent income hypothesis, where certainty equivalence holds and risk plays a role for consumption allocation, is a special case of this environment. We then move beyond certainty equivalence and analyze environments where risk matters for consumption and saving. We analyze the two sources of precautionary saving, prudence and occasionally binding borrowing constraints, and explain how they induce concavity in the consumption function. In Section 11.4, we combine a continuum of households facing income fluctuation problems, and study how they give rise to an endogenous joint distribution of income, consumption, and wealth. We then characterize the stationary equilibrium of these economies, first without and then with production. This last section extends the analysis of Section 7.6.

Because of space constraints, we omit developing a number of interesting and important topics related to consumption. For example, business cycles and asset pricing, life-cycle patterns, information frictions, consumer durables and housing, bequest, habits, and non-standard preferences. Some of these topics will be covered in later chapters, such as Chapters 14, 16 and 21. We also refer the reader to the surveys by Hall (1988), Muellbauer (1994), Browning and Lusardi (1996), Attanasio (1999), Browning and Crossley (2001), Campbell (2003), Attanasio and Weber (2010), Meghir and Pistaferri (2011), Piazzesi and Schneider (2016), Kaplan and Violante (2022), and the thematic books by Deaton (1992) and Jappelli and Pistaferri (2020) for additional material.

## 11.2 Consumption under autarky and full insurance

Consider an endowment economy with aggregate uncertainty, as the one outlined in Chapter 7 of the book. Let  $\omega_t \in \Omega$  (a finite set) be the realization of a stochastic event (e.g., an aggregate shock) at date  $t$ . Let  $\omega^t = \{\omega_0, \omega_1, \dots, \omega_t\}$  be the history of events until time  $t$ , with  $\omega^t \in \Omega^t \equiv \Omega \times \Omega \times \dots \times \Omega$ , the  $t + 1$  Cartesian product of  $\Omega$ . Each history  $\omega^t$  has unconditional probability of occurring  $\pi(\omega^t)$ . We assume all households have rational expectations, i.e. they forecast by using the true probability distribution. The economy is populated by a continuum of measure one infinitely-lived households indexed by  $i$  who are endowed with stochastic income  $y_{i,t}(\omega^t)$  such that

$$\int_0^1 y_{i,t}(\omega^t) di = Y_t(\omega^t),$$

where  $Y_t(\omega^t)$  is the (random) aggregate endowment of the economy. Each realization  $\omega_t \in \Omega$  corresponds to a particular value of the aggregate endowment and a particular distribution

of it across households.<sup>1</sup>

Households are expected utility maximizers, with period utility  $u(c_{i,t}(\omega^t))$ , where  $c_{i,t}(\omega^t)$  is consumption of individual  $i$  upon realization of history  $\omega^t$ . We assume that  $u$  satisfies standard properties, i.e.  $u' > 0$  and  $u'' < 0$ . Let  $C_t(\omega^t) = \int_0^1 c_{i,t}(\omega^t) di$  denote aggregate consumption. Aggregate feasibility implies that aggregate consumption equals the aggregate endowment along each history

$$C_t(\omega^t) = Y_t(\omega^t), \text{ for all } t, \omega^t \in \Omega^t.$$

The individual consumption allocation that arises in the equilibrium of this economy depends on market arrangements. We consider two extreme benchmarks: autarky and full insurance.

Under autarky, there is no insurance market which allows individuals to trade across states, and no storage technology to transfer resources across time (e.g., the endowment is fully perishable). In this economy, an individual  $i$  who receives a random stream of income shocks  $\{\{y_{i,t}(\omega^t)\}_{\omega^t \in \Omega^t}\}_{t=0}^\infty$  has no other choice than consuming their income in every state:

$$c_{i,t}(\omega^t) = y_{i,t}(\omega^t), \text{ for all } t, \omega^t \in \Omega^t, \quad (11.1)$$

and equation (11.1) is also their budget constraint. Under autarky there is full pass-through of individual income shocks into consumption, or no consumption smoothing whatsoever.

Consider now the other end of the spectrum of market arrangements: complete markets (also called full insurance or full risk-sharing), introduced earlier in Section 7.4. Under this arrangement, households can trade a complete set of Arrow securities. This market structure allows every individual  $i$  to achieve any transfer of income across states and across time, as long as these trades respect the time-zero Arrow-Debreu budget constraint

$$\sum_{t=0}^\infty \sum_{\omega^t \in \Omega^t} p_t(\omega^t) [c_{i,t}(\omega^t) - y_{i,t}(\omega^t)] = 0, \quad (11.2)$$

where  $p_t(\omega^t)$  is the price of the Arrow security for history  $\omega^t$ . The Arrow-Debreu competitive equilibrium for this economy was defined in Section 7.4, where it is shown that for any pair of households  $(i, j)$ ,

$$\frac{u'(c_{i,t}(\omega^t))}{u'(c_{j,t}(\omega^t))} = \frac{\lambda_i}{\lambda_j}, \text{ for all } t, \omega^t \in \Omega^t, \text{ and } (i, j), \quad (11.3)$$

with  $\lambda_i$  and  $\lambda_j$  representing Lagrange multipliers on their lifetime budget constraint.

Equation (11.3) illustrates the defining property of consumption allocations under full insurance: when households can trade a complete set of Arrow securities, *the ratio of marginal utility of consumption of any two households is constant across time and states.*

To make further analytical progress, note that in the special case of CRRA utility, where

$$u(c_{i,t}(\omega^t)) = \frac{c_{i,t}(\omega^t)^{1-\sigma}}{1-\sigma}, \text{ with } \sigma \in [0, \infty)$$

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<sup>1</sup>For example, with  $I = 2$  and  $\omega_t \in \{\omega^L, \omega^H\}$ , we could have a configuration where  $Y(\omega^L) = 2, Y(\omega^H) = 4$ , and  $y_1(\omega^L) = 2, y_2(\omega^L) = 0, y_1(\omega^H) = 1, y_2(\omega^H) = 3$ . Thus, the aggregate endowment in state  $H$  is larger than in state  $L$ , but type 1 is better off in state  $L$  and type 2 in state  $H$ .

equation (11.3) becomes

$$\frac{c_{i,t}(\omega^t)}{c_{j,t}(\omega^t)} = \left(\frac{\lambda_i}{\lambda_j}\right)^{-\frac{1}{\sigma}},$$

which implies that the ratio of consumption allocations (not just marginal utility of consumption) between households is constant across states and over time. Summing over  $i = 1, \dots, I$  on both sides yields

$$c_{j,t}(\omega^t) = \left[ \frac{(\lambda_j)^{-\frac{1}{\sigma}}}{\int_0^1 (\lambda_i)^{-\frac{1}{\sigma}} di} \right] C_t(\omega^t), \quad (11.4)$$

so, individual consumption is proportional to aggregate consumption (or aggregate endowment), with a coefficient of proportionality, in the square bracket, that depends on the relative values of  $\lambda$ .<sup>2</sup> Thus, individual consumption is not constant over time, but it is independent of the realization of the individual endowment. Unanticipated shocks to the aggregate endowment are the only source of individual variation in consumption. These fluctuations cannot be insured because, by definition, they are common across households.

### 11.2.1 Full insurance with preference heterogeneity

How does preference heterogeneity affect consumption allocations under complete markets? It is immediate to see from (11.3) that the hallmark of complete markets —constant ratio of marginal utility of consumption across households— is still valid even with heterogeneity in  $u$ . It is no longer necessarily true, however, that the ratio of consumption is also equalized. To see this, assume  $u$  is CRRA and households (indexed by  $i$ ) differ with respect to the curvature parameter  $\sigma$

$$u^i(c_{i,t}(\omega^t)) = \frac{c_{i,t}(\omega^t)^{1-\sigma_i}}{1-\sigma_i}.$$

Consider an economy with two agents  $i = 1, 2$ . The individual FOCs conditions combined with the market clearing condition imply

$$\lambda_2 c_{1,t}(\omega^t)^{-\sigma_1} = \lambda_1 [Y_t(\omega^t) - c_{1,t}(\omega^t)]^{-\sigma_2}. \quad (11.5)$$

Figure 11.1 plots right-hand side and left-hand side as a function of  $c_{1,t}$  for the case where type 1 is less risk averse than type 2 ( $\sigma_1 < \sigma_2$ ). The two curves cross only once. Now consider a rise in the aggregate endowment. We know from (11.4) that when  $\sigma_1 = \sigma_2$  consumption would increase proportionately so to leave the ratio of consumption between the two households unchanged. With unequal risk aversion, instead, it is efficient for the planner to have the consumption of the least risk-averse households (type 1 in our example) fluctuate more in response to aggregate shocks, as evident from Figure 11.1.

### 11.2.2 Empirical tests of the full insurance hypothesis

Abstracting from heterogeneity in preference for risk, one can combine the consumption allocation under autarky in (11.1) and complete markets in (11.4) to derive an encompassing

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<sup>2</sup>We have derived this result in Section 5.2.1. There, our formulation did not include uncertainty and we have derived it directly from the equilibrium conditions.

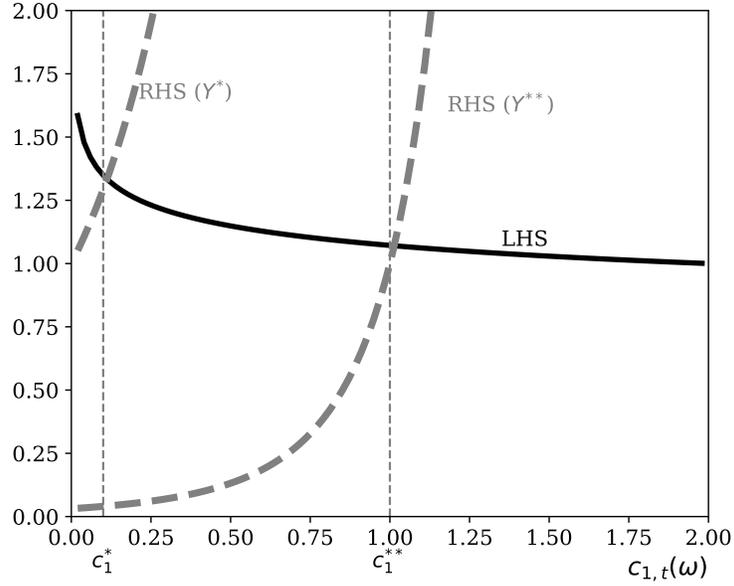


Figure 11.1: Plot of the left-hand-side (LHS) and right-hand-side (RHS) of equation (11.5).

**Notes:** Parameter values:  $\sigma_1 = 0.1, \sigma_2 = 5, \lambda_1 = \lambda_2 = 1$ . Aggregate endowment rises from  $Y^* = 1$  to  $Y^{**} = 2$ . Agent 1, who is less risk averse, absorbs most of the change in the endowment.

empirical model for consumption growth that can be taken to the data:

$$\Delta \log c_{i,t} = \beta_1 \Delta \log C_t + \beta_2 \Delta \log y_{i,t} + \varepsilon_{i,t},$$

where  $y_{i,t}$  is current individual income,  $C_t$  is aggregate consumption, and  $\varepsilon_{i,t}$  is an error term independent of the two regressors.<sup>3</sup> The autarky hypothesis implies  $\beta_1 = 0$  and  $\beta_2 = 1$ , i.e., individual consumption tracks individual income. The full risk-sharing hypothesis, instead, implies  $\beta_1 = 1$  and  $\beta_2 = 0$ , i.e. individual consumption tracks aggregate endowment, but is independent of individual income.

In the early 1990s' a large empirical literature developed with the aim of testing these hypotheses using longitudinal micro data on consumption expenditures and income (Altug and Miller, 1990, Mace, 1991, Cochrane, 1991, Nelson, 1994, Townsend, 1994, Attanasio and Davis, 1996, Hayashi, Altonji, and Kotlikoff, 1996, and Jappelli and Pistaferri, 2006). Overall, the conclusion of this literature is that an empirically plausible model of consumption behavior lies somewhere in between full risk-sharing and autarky: individual income shocks are only *partially insured*.

#### More details

Mace (1991) used the short panel dimension of the Consumer Expenditure Survey (CES), the main survey on consumption expenditures of U.S. households, to test the relation

<sup>3</sup>This equation can also be interpreted as the result of a log-linearization for more general preferences.

between individual and aggregate consumption growth, and rejected the null hypothesis that  $\beta_1 = 1$ . [Cochrane \(1991\)](#), instead, tested the null hypothesis that individual consumption does not react to idiosyncratic income fluctuations ( $\beta_2 = 0$ ). He used the Panel Study of Income Dynamics (PSID), the longest-running longitudinal dataset representative of the U.S. population, to identify explicit events associated with income losses, such as days of work lost because of illness, involuntary job loss, weeks spent by jobless household heads looking for employment, days of work lost to strikes. He concluded that household food consumption expenditures (the only spending category well measured in this dataset at the time) are quite —although not fully— responsive to many of these indicators, a finding that contradicts the efficient risk sharing hypothesis. [Attanasio and Davis \(1996\)](#) analyzed the relation between changes in consumption and changes in relative earnings across demographic groups (defined by cohorts and education) in the U.S. in the 1980s. Their results indicate a “spectacular failure” of the full insurance hypothesis across groups.

According to the full risk-sharing hypothesis under CRRA, individual consumption moves in lockstep with aggregate consumption for all households. As a result, the household ranking (i.e., their relative position) in the consumption distribution remains constant over time. The hypothesis of lack of rank mobility in household consumption was tested, and amply rejected, by [Fisher and Johnson \(2006\)](#) and [Jappelli and Pistaferri \(2006\)](#).

Tests of perfect risk sharing were extended to the case of risk aversion heterogeneity by [Mazzocco and Saini \(2012\)](#) and [Schulhofer-Wohl \(2011\)](#). As discussed by these authors, preference heterogeneity can lead to a spurious rejection of this hypothesis. To see this, note that, as explained above, if households differ in their degree of curvature in utility, optimality conditions for the CRRA case imply that

$$\Delta \log c_{i,t} = \beta_{i,1} \Delta \log C_t + \beta_2 \Delta \log y_{i,t} + \varepsilon_{i,t} \quad (11.6)$$

where  $\beta_{i,1}$  is larger the lower is risk aversion of household  $i$  relative to its population average. Thus, if one estimates the misspecified equation (11.4) instead of (11.6), the error includes the term  $(\beta_{i,1} - 1) \Delta \log C_t$ . If earnings of low risk aversion (high  $\beta_{i,1}$ ) individuals are more procyclical, then this omitted variable would induce a positive bias on the estimate of  $\beta_2$ . One reason to expect that less risk-averse households will have more procyclical incomes is that more risk tolerant workers will choose occupations that carry more risk, both idiosyncratic and aggregate. [Schulhofer-Wohl \(2011\)](#) uncovers some evidence about this mechanism from respondents in the Health and Retirement Study.

### 11.2.3 Two approaches to partial risk sharing

How do we go about developing and quantifying economic theories of partial risk sharing? Economists have followed two methodologies. The first one —which we may call the *endogenous incomplete markets* approach— is rooted in the tradition of microfoundation of macroeconomics. According to this approach, one should model explicitly the fundamental frictions that undermine the emergence of full insurance in the competitive equilibrium.

Recall that a complete set of state-contingent securities will be traded if two assumptions hold: (i) perfect enforcement of contracts and (ii) perfect information. Deviations from these assumptions, such as limited commitment or private information (adverse selection or moral hazard), lead to partial risk sharing. There exist several examples of models that incorporate these frictions in the competitive equilibrium and analyze implications for the distribution of consumption allocations. For example, building on [Kehoe and Levine \(1993, 2001\)](#), [Krueger and Perri \(2006\)](#) relax the perfect contract enforcement assumption; [Doepke and Townsend \(2006\)](#) and [Attanasio and Pavoni \(2011\)](#) relax the perfect information assumption.

The principal strength of this methodology is that the market structure is not assumed exogenously, but it emerges in equilibrium and, as a result, is endogenous with respect to changes in primitives of the economy and to government policy.<sup>4</sup> The main shortcomings of this strategy are two. First, the financial contracts that end up being traded in equilibrium are quite complex, e.g. they are history dependent and state-contingent, and thus very different from the simple ones we observe in reality. For example, in the limited enforcement economy default is an off-equilibrium threat, but it never actually happens, whereas households do default in the real world. Second, the empirical implications of these models for consumption allocations are often rejected by the data. Take, again, the limited commitment model as an example. Consumption always drifts down when the participation constraint of the household does not bind, and it jumps up when it does bind, which happens whenever income increases sufficiently ([Kocherlakota, 1996](#)). This pattern gives rise to an extreme and counterfactual degree of left-skewness in the consumption distribution ([Broer, 2013](#)). This disconnect between theory and data represents a challenge for quantitative analysis of these models.

This criticism fueled the second methodology, which we can call the *exogenous incomplete markets* approach. This alternative view prescribes that we should only model the assets that we observe in the real world, e.g. non-state-contingent bonds, risky publicly traded equity, housing, privately held firms, etc. The advantage of this perspective is that it leads to models that can be easily and naturally taken to the data. Its drawback is that the asset market structure is exogenously assumed and, as a result, it does not respond to changes in model parameters. To be precise, in an environment with exogenous incomplete markets, a shift in primitives (e.g., in the stochastic process of income shocks) does not lead to the addition or the disappearance of certain securities being traded, but it does impact the equilibrium prices at which existing assets are traded (and, possibly, also impacts the value of borrowing limits, depending on how they are specified). As a result, the pass-through of income shocks to consumption, an indicator of the degree of risk sharing, is also affected.

Before articulating the analysis of this approach, it is worth noting that there exist some results in the literature which show that the right combination of fundamental frictions can give rise to a realistic market structure emerging endogenously in equilibrium. Notable examples are [Allen \(1985\)](#) and [Cole and Kocherlakota \(2001\)](#) who consider an environment with unobservable income shocks and hidden saving, and show that the constrained efficient allocations can be decentralized through a competitive asset market where households only

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<sup>4</sup>For example, in the limited commitment economy of [Krueger and Perri \(2006\)](#), an increase in the size of idiosyncratic income risk reduces the value of autarky (and incentives to default), and increases equilibrium risk sharing, a force that contains the rise in consumption inequality caused by higher income uncertainty.

trade a non-contingent bond.<sup>5</sup>

The canonical example of the exogenous incomplete market approach is the so-called *bond economy*, an environment where households are only allowed to trade a one period non-state-contingent bond. This arrangement is reminiscent of a standard deposit/loan contract in the real world, where a saver receives an interest on their deposits every period, and a borrower repays interests on their loan every period without ever defaulting.<sup>6</sup> To fully understand the ad-hoc restrictions on market structure that we impose to obtain the bond economy, start from complete markets and consider the sequential formulation version of the Arrow-Debreu budget constraint (11.2) holding at every history  $\omega^t \in \Omega^t$  that we have studied in Section 7.4:

$$c_{i,t}(\omega^t) + \sum_{\omega_{t+1} \in \Omega} q_t(\omega_{t+1}, \omega^t) a_{i,t+1}(\omega_{t+1}, \omega^t) = y_{i,t}(\omega^t) + a_{i,t}(\omega^t) \quad (11.7)$$

where  $a(\omega_{t+1}, \omega^t)$  is an Arrow security purchased at date  $t$  and state  $\omega^t$  that pays one unit of consumption if state  $\omega_{t+1}$  occurs next period,  $q(\omega_{t+1}, \omega^t)$  is the price of such Arrow security, and  $a_{i,t}(\omega^t)$  are all the Arrow securities purchased at  $t-1$  which pay in the current realized state  $\omega^t$ . In the bond economy, we force agents to trade only a non-state-contingent asset. The budget constraint (11.7) is therefore replaced by the more restrictive

$$c_{i,t}(\omega^t) + q_t(\omega^t) a_{i,t+1}(\omega^t) = y_{i,t}(\omega^t) + a_{i,t}(\omega^{t-1}), \quad (11.8)$$

where  $a_{t+1}(\omega^t)$  is a bond purchased at date  $t$  and state  $\omega^t$  that pays one unit of consumption next period, independently of the realization of the state  $\omega_{t+1}$ , and  $q_t(\omega^t)$  is the price of such bond.

As the terminal condition, for now we only impose the no-Ponzi game (nPg) condition and do not impose any further borrowing constraints. The nPg condition for this market structure is

$$\lim_{t \rightarrow \infty} \left( \prod_{s=0}^t q_s(\omega^s(\omega^t)) \right) a_{i,t+1}(\omega^t) \geq 0 \quad (11.9)$$

for all  $\omega^t \in \Omega^t$ , where  $\omega^s(\omega^t)$  represents the sub-history of  $\omega^t$  up to period  $s$ . A similar condition (for the deterministic case) has shown up in Section 5.3.1. Optimality implies the transversality condition (TVC)  $\lim_{t \rightarrow \infty} \left( \prod_{s=0}^t q_s(\omega^s(\omega^t)) \right) a_{i,t+1}(\omega^t) \leq 0$  for all  $\omega^t \in \Omega^t$ , since a household with a positive limiting value of their wealth can improve their welfare by dissaving and consuming a bit extra.<sup>7</sup> Combining these two inequalities, we obtain the condition

$$\lim_{t \rightarrow \infty} \left( \prod_{s=0}^t q_s(\omega^s(\omega^t)) \right) a_{i,t+1}(\omega^t) = 0 \quad \text{for all } \omega^t \in \Omega^t. \quad (11.10)$$

To simplify the analysis, in what follows we assume away fluctuations in the aggregate endowment  $Y_t(\omega^t)$  which implies that the bond price is a constant,  $q_t(\omega^t) = q$  and that we

<sup>5</sup>Broer, Kapička, and Klein (2017) obtain similar implications from an economy which combines limited enforcement of contracts and private information about earnings.

<sup>6</sup>In reality such contracts are intermediated by banks. To the extent that the financial sector is competitive and banks solve a static maximization problem, financial intermediaries play no interesting role in the model and can be ignored. Chapter 19 discusses dynamic models of financial frictions where banks play a crucial role in determining equilibrium allocations.

<sup>7</sup>These concepts were introduced in Section 4.3.1.

can omit the dependence on histories. Since the bond pays one unit of consumption in the next period, its rate of return is  $r = 1/q - 1$ . We can therefore reformulate the individual budget constraint of the bond economy (11.8) as

$$a_{i,t+1} = (1 + r)(y_{i,t} + a_{i,t} - c_{i,t}). \quad (11.11)$$

This budget constraint (which reflects the market structure of the bond economy) is one of the cornerstones of the analysis of consumption and saving behavior in modern macroeconomics.<sup>8</sup> In this economy there are no explicit insurance markets, but, beyond borrowing when allowed, saving and dissaving act as a mechanism for *self-insurance*.<sup>9</sup> Throughout the rest of the chapter, we'll maintain this market structure.

## 11.3 Income fluctuation problems

We begin with a partial equilibrium analysis of the intertemporal decision problem of an infinitely-lived household who is subject to fluctuations in labor income, and every period must decide how much to consume and how much to save in a risk-free non-state contingent asset, possibly subject to a borrowing limit. The household takes the interest rate as given.

We analyze this *income fluctuation problem* (in the language of [Schechtman and Escudero, 1977](#)) first in the deterministic case and then in the stochastic case. A special case of this model, when the utility function is quadratic, is [Hall \(1978\)](#)'s formulation of [Friedman \(1957\)](#)'s permanent income hypothesis which displays certainty equivalence. We then generalize the model to settings where risk matters, either because of the presence of binding liquidity constraints or because the utility function displays prudence. For each of these models, we characterize the marginal propensity to consume.

### 11.3.1 Deterministic case

We begin by abstracting from income uncertainty. Consider the problem of an individual who faces deterministic (i.e., perfectly known ex-ante) income fluctuations  $\{y_t\}_{t=0}^{\infty}$ , has to choose optimally how to allocate consumption  $c_t$  over time, and can only save through a risk free bond  $a_t$ . This individual solves

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad (11.12)$$

subject to

$$a_{t+1} = (1 + r)(y_t + a_t - c_t).$$

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<sup>8</sup>Note the timing convention implicit in the way we wrote this budget constraint: income is paid and consumption is chosen at the beginning of the period, and thus interests accrue on savings defined as  $y_t + a_t - c_t$ . The alternative timing convention, which we'll sometimes use in this chapter, is that income is paid and consumption decisions are made at the end of the period, which leads to the formulation of the budget constraint:  $a_{t+1} = y_t + (1 + r)a_t - c_t$ .

<sup>9</sup>The expression "self-insurance" refers to the fact that individuals are insuring against future shocks by dissaving and saving, i.e. by trading intertemporally with themselves, and not by trading state-contingent insurance contracts with others, as such contracts are not available in this market structure.

This problem is analogous to the consumption-saving model from Chapter 4, but extended to incorporate time-varying deterministic income levels  $y_t$ .

By solving the Lagrangian associated to this problem, it is easy to see that the household first-order condition yields

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 + r. \quad (11.13)$$

This *consumption Euler equation* has the standard interpretation: the marginal rate of substitution between consumption today and consumption next period (intended as two different goods) equals the price of consumption today relative to the price of consumption next period, or the interest rate.<sup>10</sup> It also has a variational interpretation: the value of a unit of consumption today is its marginal utility. Shifting such unit to next period yields  $(1 + r)$  units valued at the discounted marginal utility tomorrow. Optimality requires the household to be indifferent, hence the equal sign.

Because of the strict concavity of  $u$ , the Euler equation implies that the slope of the optimal individual consumption profile between  $t$  and  $t + 1$  is increasing in  $\beta$  and in  $r$ . Both higher patience and higher return on saving induce the household to save more today and postpone consumption to the future, which tilts upward the consumption profile.<sup>11</sup> Turning to the special case of CRRA utility with curvature parameter  $\sigma$ , (11.13) becomes

$$\frac{c_{t+1}}{c_t} = [\beta(1 + r)]^{\frac{1}{\sigma}}. \quad (11.14)$$

This formulation clarifies that the extent to which variation in  $\beta$  and  $r$  translates into steeper or flatter consumption paths depends on the elasticity of intertemporal substitution,  $EIS = 1/\sigma$ . See Section 4.2.4 for details on the derivation of the *EIS*.

To sum up, in this environment, households want to smooth consumption with respect to deterministic income fluctuations, and thus they save when income is high relative to its mean and dissave when it is low. The extent of consumption smoothing (i.e., how close  $c_t$  and  $c_{t+1}$  are to each other) depends on the product  $\beta(1 + r)$  and on the willingness of households to substitute intertemporally, determined by  $1/\sigma$ . The closer  $\beta(1 + r)$  is to 1 and the lower the elasticity of substitution, the more consumption is smoothed across periods.

### Marginal propensities to consume

Let  $R \equiv 1 + r$  and iterate forward the budget constraint to obtain, after imposing condition (11.10)

$$c_t + \frac{1}{R}c_{t+1} + \frac{1}{R^2}c_{t+2} + \dots = a_t + \sum_{j=0}^{\infty} \left(\frac{1}{R}\right)^j y_{t+j}.$$

<sup>10</sup>To see why the interest rate is the intertemporal price of consumption, rewrite the budget constraint (11.8) in nominal terms with the price of the final good multiplying quantities at  $t$  and  $t + 1$

$$q_{t+1}a_{t+1} = q_t a_t + q_t y_t - q_t c_t,$$

then divide through by  $q_{t+1}$  and compare with (11.8).

<sup>11</sup>These statements are about the relative consumption across the two periods. Whether the *level* of consumption at  $t$  increases or decreases depends on the relative strength of income and substitution effect due to the increase in  $r$  (and thus on the level of wealth  $a_t$ ), as for any other change in relative prices.

Using the Euler equation (11.14) to substitute  $c_{t+j}$  for all  $j > 0$  on the left hand side as a function of  $c_t$ , and collecting terms, we arrive at:

$$c_t = \left(1 - R^{-1} (\beta R)^{\frac{1}{\sigma}}\right) \left[ a_t + \sum_{j=0}^{\infty} \left(\frac{1}{R}\right)^j y_{t+j} \right]. \quad (11.15)$$

This expression is useful to introduce the concept of *marginal propensity to consume* (MPC). The MPC out of wealth  $a_t$  (or, equivalently, out of a transitory change in income  $y_t$ ) is defined as  $\partial c_t / \partial a_t$ .<sup>12</sup> Differentiating (11.15), we obtain that

$$MPC = 1 - R^{-1} (\beta R)^{\frac{1}{\sigma}}. \quad (11.16)$$

Two special cases are of interest. First, recall that in the equilibrium of a representative agent model without growth,  $\beta R = 1$  and thus  $MPC = 1 - \beta$ . Let  $\beta \equiv 1/(1 + \rho)$ , where  $\rho$  is the discount rate. Since  $\rho$  is small relative to 1,  $MPC \simeq \rho$ .<sup>13</sup> Thus, in the representative agent model, the marginal propensity to consume is approximately equal to the discount rate. Second, assuming log-utility ( $\sigma = 1$ ) in the individual problem, without imposing  $\beta R = 1$ , by following similar steps one obtains  $MPC \simeq r$ .

Expression (11.15) shows that in this simple model optimal consumption is linear in wealth. In addition, equation (11.15) is an incarnation of Friedman's permanent income hypothesis: the term in the square bracket is the sum of financial wealth  $a_t$  plus human wealth (the present value of future income), i.e. total wealth. Optimal consumption equals a constant fraction of total wealth. This equation also illustrates one of the key concepts in Friedman's theory of consumption: the marginal propensity to consume out of transitory and permanent changes in income are different. Consider the case  $\beta R = 1$  where the MPC out of transitory income is  $1 - R^{-1}$ . From (11.15), it is easy to see that a permanent change in income of one unit, i.e. an increase of one unit in  $y_{t+j}$  for all  $j \geq 0$ , leads to a change in human wealth equal to  $1/(1 - R^{-1})$  and, thus a change in consumption equal to exactly 1 unit. Thus, the MPC out of permanent income is 1, and much larger than the MPC out of transitory income. Next, we explore the permanent income hypothesis in more detail.

### 11.3.2 Permanent income hypothesis

We now reintroduce income uncertainty and rewrite the household problem with the conditional expectation operator in the objective function as

$$\max_{\{c_t\}_{t=0}^{\infty}} \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t u(c_t) \quad (11.17)$$

subject to

$$a_{t+1} = (1 + r)(y_t + a_t - c_t).$$

We also make two additional assumptions: quadratic utility

$$u(c_t) = b_1 c_t - \frac{1}{2} b_2 c_t^2, \quad b_2 > 0, \quad c_t < b_1/b_2,$$

<sup>12</sup>By transitory we mean a change in  $y_t$  which leaves unchanged income  $y_{t+j}$  at any  $j > 0$ .

<sup>13</sup>Another way of deriving this result is to define  $\beta = \exp(-\rho)$  and use the approximation  $\exp(x) \simeq 1 + x$ .

and  $\beta R = 1$ . From the consumption Euler equation implied by (11.17), jointly with these two assumptions, we obtain

$$c_t = \mathbb{E}_t c_{t+1}. \quad (11.18)$$

This is the well known result of Hall (1978) that consumption is a martingale (or a random walk).<sup>14</sup> From the law of iterated expectations and the martingale property of the optimal consumption allocation:

$$\mathbb{E}_t c_{t+2} = \mathbb{E}_t [\mathbb{E}_{t+1} c_{t+2}] = \mathbb{E}_t c_{t+1} = c_t$$

and, more in general:

$$\mathbb{E}_t c_{t+j} = c_t, \text{ for any } j \geq 0. \quad (11.19)$$

If we iterate forward  $J$  times on budget constraint in (11.17), and apply the conditional expectations to deal with uncertain future realizations of income and consumption, we obtain

$$\sum_{j=0}^J \left(\frac{1}{R}\right)^j \mathbb{E}_t c_{t+j} = a_t + \sum_{j=0}^J \left(\frac{1}{R}\right)^j \mathbb{E}_t y_{t+j} - \left(\frac{1}{R}\right)^{J+1} \mathbb{E}_t a_{t+J+1}$$

Taking the limit as  $J \rightarrow \infty$  and using condition (11.10), the last term goes to zero. Using the martingale property (11.19) into the left hand side, we obtain

$$c_t = \frac{r}{1+r} \left[ a_t + \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j \mathbb{E}_t y_{t+j} \right]. \quad (11.20)$$

This expression illustrates the stochastic version of the permanent income hypothesis: optimal consumption is, again, linear and equals the annuity value of total wealth, i.e. financial wealth plus human wealth.<sup>15</sup>

If one assumes that income is deterministic and solves (11.17), one obtains  $c_{t+1} = c_t$  from the Euler equation and, by iterating forward on the budget constraint,

$$c_t = \frac{r}{1+r} \left[ a_t + \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j y_{t+j} \right] \quad (11.21)$$

Comparing (11.21) to (11.20) demonstrates that consumption satisfies *certainty equivalence*: in order to obtain the solution of the stochastic problem (11.17), it suffices solving the deterministic problem and applying conditional expectations to the exogenous variables  $\{y_{t+j}\}_{j=0}^{\infty}$  in place of the variables themselves. Put differently, no higher moment of the income process, beyond the mean, matters for the dynamics of consumption. This property

<sup>14</sup>A stochastic process  $\{x_t\}$  is a random walk when it satisfies, at every  $t$ ,  $\mathbb{E}_t x_{t+j} = x_t$  for any  $j > 0$ .

<sup>15</sup>The annuity value is exactly  $r/(1+r)$ , i.e. that portion of wealth that, when consumed every period, keeps asset holdings constant. To see this, abstract from income  $y_t$ , and note that

$$a_{t+1} = (1+r)(a_t - c_t) = (1+r) \left( a_t - \frac{r}{1+r} a_t \right) = a_t.$$

descends directly from the linear-quadratic structure of the problem. Any deviation from quadratic objective and linear constraints breaks certainty equivalence.

From (11.20), the change in consumption at time  $t$  equals

$$\Delta c_t = c_t - c_{t-1} = c_t - \mathbb{E}_{t-1}c_t = \frac{r}{1+r} [\varpi_t - \mathbb{E}_{t-1}\varpi_t], \quad (11.22)$$

where  $\varpi_t$  is total wealth defined as

$$\varpi_t \equiv a_t + \sum_{j=0}^{\infty} \left( \frac{1}{1+r} \right)^j \mathbb{E}_t y_{t+j},$$

and the last term on the right hand side is the innovation, or unexpected change, in permanent income at time  $t$ , i.e. the difference between realization and conditional expectation

$$\begin{aligned} \varpi_t - \mathbb{E}_{t-1}\varpi_t &= a_t - \mathbb{E}_{t-1}a_t + \sum_{j=0}^{\infty} \left( \frac{1}{1+r} \right)^j [\mathbb{E}_t y_{t+j} - \mathbb{E}_{t-1}(\mathbb{E}_t y_{t+j})] \\ &= \sum_{j=0}^{\infty} \left( \frac{1}{1+r} \right)^j (\mathbb{E}_t - \mathbb{E}_{t-1}) y_{t+j}, \end{aligned} \quad (11.23)$$

where we have used the law of iterated expectations  $\mathbb{E}_{t-1}(\mathbb{E}_t y_{t+j}) = \mathbb{E}_{t-1} y_{t+j}$ , and the fact that  $a_t = \mathbb{E}_{t-1}a_t$ , since there is no uncertainty at time  $t$  (after  $y_t$  is realized) about the evolution of wealth into  $t+1$ . Combining (11.22) and (11.23), we arrive at

$$\Delta c_t = \frac{r}{1+r} \sum_{j=0}^{\infty} \left( \frac{1}{1+r} \right)^j (\mathbb{E}_t - \mathbb{E}_{t-1}) y_{t+j}. \quad (11.24)$$

This equation contains another useful result: under the permanent income hypothesis, the change in consumption between  $t-1$  and  $t$  is proportional to the revision in expected future income due to the new information (the “news”) accruing in that same time interval.

### Permanent and transitory income shocks

To make further progress, we need to make some assumptions on the income process. We choose a specification that is very common in labor economics, at least since [Abowd and Card \(1989\)](#). We model labor income as the sum of two orthogonal components,  $y_t^p$  which follows a martingale with i.i.d. innovation (or shocks)  $v_t$ , and  $u_t$  which is also an i.i.d. shock

$$\begin{aligned} y_t &= y_t^p + u_t, \\ y_t^p &= y_{t-1}^p + v_t. \end{aligned} \quad (11.25)$$

In addition, we assume that  $\mathbb{E}(v_t) = \mathbb{E}(u_t) = 0$  and that the two shocks are orthogonal,  $u_t \perp v_\tau$  for all pairs  $(t, \tau)$ . If we let  $x_t$  denote either shock, our assumptions imply that  $\mathbb{E}_t(x_{t-j}) = x_{t-j}$  for  $j \geq 0$ , and  $\mathbb{E}_t(x_{t+j}) = 0$  for  $j > 0$ .

Combining these two equations in (11.25), we obtain the representation

$$y_t = y_{t-1} + u_t - u_{t-1} + v_t. \quad (11.26)$$

Using this income process into (11.24), after some algebra, we obtain

$$\Delta c_t = \frac{r}{1+r} u_t + v_t, \quad (11.27)$$

which establishes that households adjust their consumption responding to the annuity value of transitory shocks and to the full value of permanent shocks. This finding is analogous to the one for the deterministic economy: the pass-through of income shocks to consumption depends on their expected duration. Remarkably, this version of the bond economy is quite close to full insurance with respect to transitory shocks. In conclusion, borrowing and saving through a non-state contingent asset offers ample opportunity for consumption smoothing as long as shocks are not too persistent.

### Using the PIH to learn about the nature of the rise in inequality

Income inequality increased substantially in the 1980s and the 1990s in many developed countries. Much of the public, including many policymakers and economists, interpreted this trend as indicating widening differentials in standard of living across households. This interpretation, however, is open to the criticism that current income may not reflect the long-run level of resources available to a household, and hence their welfare.

Blundell and Preston (1998) showed how one can use the permanent income hypothesis, together with data on the joint cross-sectional distribution of income and consumption, to learn whether the rise in cross-sectional income inequality is of a transitory nature and hence not too worrisome, or of a permanent nature and thus detrimental for inequality in household welfare.

Consider the model of Section 11.3.2. From equations (11.26) and (11.27), we have that the evolution of income and optimal consumption for an individual  $i$  at time  $t$  are given by

$$\begin{aligned} y_{i,t} &= y_{i,t-1} + u_{i,t} - u_{i,t-1} + v_{i,t} \\ c_{i,t} &= c_{i,t-1} + \frac{r}{1+r} u_{i,t} + v_{i,t}. \end{aligned}$$

Now, compute the cross-sectional variance of consumption and the cross-sectional covariance between consumption and income for all individuals belonging to a cohort  $k$ , assuming that  $r \simeq 0$ . Then, one obtains:

$$\Delta \text{Var}_{k,t}(c) = \Delta \text{Cov}_{k,t}(c, y) \simeq \text{Var}_t(v). \quad (11.28)$$

In other words, by tracing the change over time of the within-cohort variance of consumption, or covariance between consumption and income, one can estimate the change over time in the variance of the permanent component of income. Blundell and Preston concluded that the bulk of the rise in UK income inequality was driven by the permanent component, a result that is consistent with the idea that skill-biased technical change (and rising college wage premium), is a key driving force of the recent changes

in income distribution. We discuss skill-biased technical change in Section 21.2.2. Although based on a different methodology, these empirical findings are reminiscent of those in [Attanasio and Davis \(1996\)](#) discussed in Section 11.2.2. In particular, they also represent a rejection of full insurance since, from the perspective of the efficient risk sharing hypothesis, consumption should not react to idiosyncratic income shocks, no matter their persistence.

### Saving for the rainy days

Define household savings  $s_t$  as capital income plus labor income net of consumption expenditures

$$s_t = \frac{r}{1+r}a_t + y_t - c_t. \quad (11.29)$$

Combining (11.29) with (11.20) we obtain an expression for saving only as a function of current and future expected income

$$s_t = y_t - \frac{r}{1+r} \sum_{j=0}^{\infty} \left( \frac{1}{1+r} \right)^j \mathbb{E}_t y_{t+j}.$$

Unfolding this summation on the right hand side, we obtain

$$\begin{aligned} s_t &= y_t - \frac{r}{1+r}y_t - \frac{r}{1+r} \left[ \left( \frac{1}{1+r} \right) \mathbb{E}_t y_{t+1} + \left( \frac{1}{1+r} \right)^2 \mathbb{E}_t y_{t+2} + \dots \right] \\ &= \frac{1}{1+r}y_t - \frac{r}{1+r} \left( \frac{1}{1+r} \right) \mathbb{E}_t y_{t+1} - \frac{r}{1+r} \left[ \left( \frac{1}{1+r} \right)^2 \mathbb{E}_t y_{t+2} + \dots \right] \\ &= -\frac{1}{1+r} \mathbb{E}_t \Delta y_{t+1} + \left( \frac{1}{1+r} \right)^2 \mathbb{E}_t y_{t+1} - \frac{r}{1+r} \left[ \left( \frac{1}{1+r} \right)^2 \mathbb{E}_t y_{t+2} + \dots \right] \end{aligned}$$

and, using the same approach on terms  $t+j$  with  $j > 1$  we obtain

$$s_t = - \sum_{j=1}^{\infty} \left( \frac{1}{1+r} \right)^j \mathbb{E}_t \Delta y_{t+j}.$$

This expression shows that savings are equal to the discounted sum of expected declines in income. If a household expects income to fall in the future, in anticipation they will save and these savings will allow them to smooth consumption once income actually declines. For example, under a mean reverting income process, households who experience a sequence of positive (above mean) income shocks will save because they understand that this abundance is only transient. Similarly, those who experience a sequence of negative (below mean) income shocks will dissave. [Campbell \(1987\)](#) calls this behavior *saving for the rainy day*. Or, dissaving in the expectation of a sunny day.

### 11.3.3 Borrowing constraints

To derive the results in the previous section, we have abstracted from borrowing constraints. But, to what extent can we safely ignore borrowing limits, as if they were never binding? The answer depends on the income process.

Suppose we impose the ad hoc no-borrowing constraint  $a_{t+1} \geq 0$  on the household problem. When income  $y_t$  is a random walk, from (11.20) we obtain

$$c_t = \frac{r}{1+r} a_t + y_t$$

since  $\mathbb{E}_t y_{t+j} = y_t$  for all  $j \geq 0$ . Substituting this expression for consumption into the budget constraint yields  $a_{t+1} = a_t$ . In this case, wealth is constant, so if a household starts with a positive wealth level, the zero debt limit will never bind.

Suppose now that  $y_t$  follows an i.i.d. shock with mean  $\bar{y}$ . From (11.20) we have

$$c_t = \frac{\bar{y}}{1+r} + \frac{r}{1+r} (a_t + y_t) \quad (11.30)$$

since  $\mathbb{E}_t y_{t+j} = \bar{y}$ . Substituting into the budget constraints yields  $\Delta a_{t+1} = y_t - \bar{y}$ . In this case, wealth follows a random walk without drift, which means that starting from any initial level of wealth there is always a positive probability that at some point in the future the household will hit the borrowing limit in finite time. For example, consider an individual who receives an income realization equal to  $y_t < (1-r)\bar{y}$  and whose initial wealth is  $a_t < r\bar{y}$ . It is easy to see that if the individual wanted to consume its optimal unconstrained level in equation (11.30), they would enter the next period with negative wealth, which would violate the no-borrowing limit. As a result, the individual would have to consume below its unconstrained level. More broadly, the borrowing constraint is likely to bind whenever  $a_t$  is small,  $y_t$  follows a mean-reverting process, and the individual is hit by a low enough shock (relative to its mean). Because income is expected to revert back to its higher mean, consumption smoothing dictates that the household should dissave and keep their consumption high. In some states, however, dissaving is not enough, borrowing would be necessary and the constraint binds. In this case, the optimal consumption choice is constrained.

These examples highlight the fact that, in general, borrowing limits cannot be ignored and their presence affects optimal consumption and saving choices, as we explain next. We now examine the more general consumption-saving problem with stochastic income  $y_t$  and an ad hoc zero-borrowing constraint.

Consider the problem of a household facing a no-borrowing constraint. Stated in a sequential formulation, we have

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \quad (11.31)$$

subject to

$$a_{t+1} = R(a_t + y_t - c_t)$$

and

$$a_{t+1} \geq 0,$$

where  $u' > 0$  and  $u'' < 0$ . The first-order condition of this problem yields the modified Euler equation:

$$u'(c_t) = \beta R \mathbb{E}_t [u'(c_{t+1})] + \lambda_t,$$

where  $\lambda_t$  is the multiplier on the borrowing constraint at date  $t$ . Because  $\lambda_t \geq 0$ , the Euler equation can be rewritten as

$$u'(c_t) \geq \beta R E_t [u'(c_{t+1})], \quad (11.32)$$

where the strict inequality holds when the constraint is binding. Assume  $\beta R = 1$ . If the constraint is not binding, the household's choice is interior, dictated by the Euler equation which states that marginal utility today is equated to expected marginal utility next period. If, instead, the constraint is binding, current marginal utility is higher and consumption is lower: the household problem has a corner solution and is not determined by the Euler equation, but by the borrowing limit and the budget constraint. Since  $a_{t+1} = 0$ , from the budget constraint, we obtain  $c_t = a_t + y_t$ , i.e. the household consumes all their cash in hand. The multiplier  $\lambda_t$  plays a role akin to the interest rate, in that it increases the cost of consuming today relative to postponing consumption to the future (Hall, 1988).

There are two key differences between the model without and with borrowing constraints. The first one is that, absent borrowing constraints, what matters for current consumption is only the expected discounted value of labor income, not its timing (see equation (11.20)). In the presence of borrowing constraints, instead, the timing of income matters for consumption. Consider two random income streams with the same discounted expected value, one that is decreasing and another one that is increasing: a household facing the latter is more likely to be constrained and their consumption path may differ from the consumption path of a household facing the decreasing stream. The second difference is that, for a constrained household, the marginal propensity to consume is 1 for both transitory and permanent shocks: no matter what the duration of the shock is, a small change in current income increases current consumption one for one. Thus constrained (or hand-to-mouth) households display higher MPC out of transitory income relative to unconstrained ones.

### Natural borrowing limit

So far, when modelling borrowing constraints we have specified ad hoc limits of the type  $a_{t+1} \geq -a$ . Another type of constraint is the so called “natural” borrowing limit. In Section 4.3.1, we discussed the notion of the natural borrowing limit in the context of a deterministic model. In the Online Appendix to Chapter 4, we have shown that, in the deterministic model, the natural borrowing limit is equivalent to the no-Ponzi-game (nPg) condition. In what follows, we extend this notion to an environment with uncertainty. Here, the natural borrowing limit is the lowest asset position consistent with non-negative consumption in every state and every period.

Start from the case where  $\{y_t\}_{t=0}^{\infty}$  is deterministic. Iterating forward on the budget constraint and imposing  $c_t \geq 0$  for all  $t$ , we obtain

$$a_{t+1} \geq - \sum_{j=0}^{\infty} \left( \frac{1}{1+r} \right)^j y_{t+1+j}.$$

Assume now that income is stochastic and the lowest possible realization is  $y_{\min}$ , always occurring with positive probability. Then, to guarantee positive consumption at every  $t$ , debt cannot exceed the present value of  $y_{\min}$ , and thus the natural borrowing limit becomes

$$a_{t+1} \geq - \left( \frac{1+r}{r} \right) y_{\min}.$$

Borrowing up to this value, or more, would mean that there is a positive probability that a state occurs where consumption must be zero. This statement is easy to prove. Suppose the household has borrowed exactly up to this limit, i.e.  $a_t = -\left(\frac{1+r}{r}\right) y_{\min}$  and they are hit by an income realization  $y_t = y_{\min}$ . Then from the budget constraint:

$$c_t = \underbrace{-\left(\frac{1+r}{r}\right) y_{\min}}_{a_t} + y_{\min} - \frac{a_{t+1}}{1+r} \leq -\frac{1}{r} y_{\min} - \frac{1}{1+r} \underbrace{\left[-\left(\frac{1+r}{r}\right) y_{\min}\right]}_{\text{max that can be borrowed}} = 0.$$

It follows that, if the utility function satisfies the Inada condition  $u(0) = -\infty$ , an optimizing consumer will never borrow up to the natural borrowing limit because doing that is inconsistent with expected utility maximization. Thus, one can always safely assume an interior solution for the Euler equation. This is, instead, not true for ad hoc debt limits which always affect the optimal solution in some region of the state space, as explained above. Finally, note that if  $y_{\min} = 0$ , then the no-borrowing constraint is also the natural borrowing limit.<sup>16</sup>

### 11.3.4 Precautionary saving

The term *precautionary saving* refers the additional amount of saving a household chooses in response to a rise in the uncertainty about future income. The certainty equivalence property of consumption allocation under quadratic utility implies that the consumption function is linear in wealth and in the expected present value of future income. Thus, a mean preserving spread of the income shock distribution (i.e., an increase in dispersion around the same mean) does not impact saving: no precautionary motive for saving is present in this environment.

In what follows we establish two forces that lead to precautionary saving behavior. The first one is the presence of occasionally binding borrowing constraints, and the second is prudence. Prudence is a property of the utility function that mathematically corresponds to a positive third derivative, i.e.  $u''' > 0$ . Under both forces a rise in income uncertainty leads to a rise in the level of savings. As first uncovered by Zeldes (1989) through numerical simulations, in these settings the consumption policy function is concave in wealth, a property that has important implications for the marginal propensity to consume.

#### Precautionary motive with borrowing constraints

To isolate the role of borrowing constraints for precautionary saving, consider again the quadratic specification for period utility, which does not display prudence because  $u''' = 0$ . Continue assuming that  $\beta R = 1$ . Suppose households face a borrowing limit  $a_{t+1} \geq -\underline{a}$ . Then,

$$c_t = \begin{cases} \mathbb{E}_t c_{t+1} & \text{if } a_{t+1} > -\underline{a} \\ y_t + a_t + \frac{\underline{a}}{R} & \text{if } a_{t+1} = -\underline{a}. \end{cases}$$

---

<sup>16</sup>Throughout this discussion, we have assumed that  $r > 0$ . As we will see in Section 11.4,  $r < 0$  can be an equilibrium outcome of economies populated by a continuum of households facing an income fluctuation problem. In such a case, it is convenient (and realistic) to assume the existence of a wedge  $\chi > 0$  between the lending rate  $r$  and the borrowing rate  $r^b$  which can be interpreted as a linear intermediation cost of the financial sector. The interest rate entering the natural borrowing limit is  $r^b = r + \chi$ , and if  $\chi$  is large enough,  $r^b > 0$  and the natural borrowing limit is well defined again.

The first line describes the optimal unconstrained intertemporal consumption allocation when the constraint is not binding, and the second line the consumption allocation if the constraint is binding, in which case the individual consumes all their resources. The above pair of conditions can be written in compound form as

$$c_t = \min \left\{ y_t + a_t + \frac{a}{R}, \mathbb{E}_t c_{t+1} \right\}. \quad (11.33)$$

Assume the constraint is not already binding at date  $t$ . Then

$$c_t = \mathbb{E}_t c_{t+1} = \mathbb{E}_t \left[ \min \left\{ y_{t+1} + a_{t+1} + \frac{a}{R}, \mathbb{E}_{t+1} c_{t+2} \right\} \right]. \quad (11.34)$$

Suppose that the uncertainty about  $y_{t+1}$  increases but its mean does not change. Very low realizations of income  $y_{t+1}$  become more likely, and thus the borrowing constraint is also more likely to bind next period. This reduces the value of  $\mathbb{E}_t \left[ \min \left\{ y_{t+1} + a_{t+1} + \frac{a}{R}, \mathbb{E}_{t+1} c_{t+2} \right\} \right]$  and of current consumption  $c_t$ . As a result, savings increase. Intuitively, when households face borrowing limits which can potentially bind in the future, they understand that if they were to receive bad income realizations, they would be pushed towards the constraint and be forced to consume their income without the ability of smoothing consumption, which reduces their welfare. To prevent this scenario, they save more. This is the first source of precautionary saving motive.

An implication of the presence of borrowing constraints is that the consumption function is concave in wealth, as illustrated in Figure 11.2.

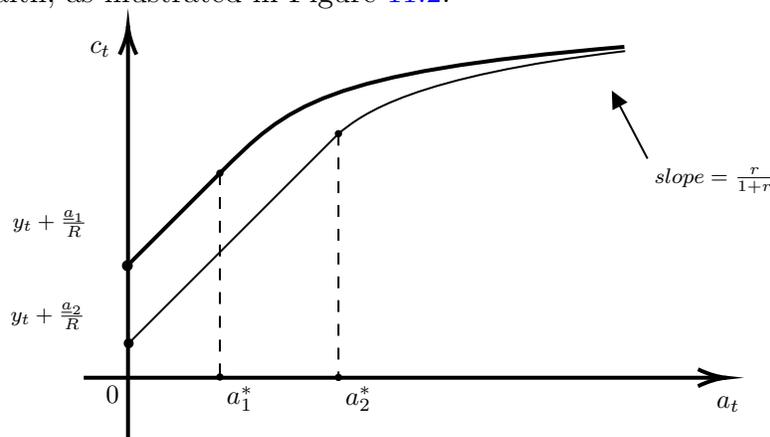


Figure 11.2: Decision rule for consumption in the presence of a borrowing constraint  $\underline{a}$  as a function of wealth  $a_t$ , for a given realization of income  $y_t$ .

**Notes:** The function is linear, with slope equal to 1, until  $a^*$ , after which it becomes concave. For large enough wealth, its slope converges to  $\frac{r}{1+r}$ . The two lines correspond to different values of the borrowing limit.

If we fix labor income  $y_t$  to a sufficiently low value, it is clear from (11.33) that there exists a level of wealth  $a_t = a^*$  below which the constraint binds and  $c_t = a_t + y_t + \frac{a}{R}$ . In this region, the slope of the consumption function with respect to wealth is 1. For sufficiently high levels of  $a_t$ , it becomes extremely unlikely (or impossible, depending on the process for  $y_t$ ) that the

constraint will bind in the future and consumption asymptotes to the linear unconstrained solution (11.20) with slope  $r/(1+r) < 1$ . For intermediate levels of wealth, the borrowing limit could be binding again, and consumption is depressed by the precautionary motive relative to the unconstrained optimum. In this range, the consumption function is strictly concave and the marginal propensity to consume is bracketed between  $r/(1+r)$  and 1.

The figure also illustrates what happens when the credit limit tightens from  $\underline{a}_1$  to  $\underline{a}_2 < \underline{a}_1$ . Obviously, households are constrained for a wider range of  $a_t$ . For unconstrained households, consumption falls across the board because of the stronger precautionary saving motive, and their marginal propensity to consume (the slope of the consumption function) increases. In the limit as wealth keeps growing, the two consumption functions converge because the borrowing constraint becomes irrelevant.

An important implication of this discussion is that there is a range of wealth levels for which the individual MPC is larger than in the PIH, either because the constraint binds (and the consumption function has slope equal to 1) or because it might bind with some probability in the future (and the consumption function is strictly concave). We return to this observation in Section 11.4.2.

### Precautionary motive with prudence

To illustrate the second source of precautionary saving, consider a simple two-period consumption-saving problem (Leland, 1968; Sandmo, 1970)

$$\max_{\{c_0, c_1, a_1\}} u(c_0) + \beta \mathbb{E}_0[u(c_1)]$$

subject to

$$c_0 + a_1 = y_0$$

and

$$c_1 = Ra_1 + y_1,$$

where  $y_0$  is given, and  $y_1$  is stochastic. Again, to isolate this second force from the first one we just described, we assume away any borrowing constraint at  $t = 0$ . The Euler equation for this problem is

$$u'(y_0 - a_1) = \beta R \mathbb{E}_0[u'(Ra_1 + y_1)], \quad (11.35)$$

an equation in one unknown,  $a_1$ . The left-hand-side is increasing in  $a_1$  since  $u'' < 0$ , and the right-hand-side is decreasing for the same reason, hence the solution for  $a_1$  is uniquely determined. What happens to optimal consumption at  $t = 0$  as future income  $y_1$  becomes more risky? Consider a mean-preserving spread of  $y_1$ . If  $u'$  is convex then, by Jensen's inequality, the value of the right-hand-side will increase. Graphically (see Figure 11.3), the left-hand-side is unchanged and the right-hand-side shifts upward, inducing a rise in optimal savings  $a_1$ .<sup>17</sup> One way to understand this result is that the hike in future consumption uncertainty reduces welfare. Increasing savings today raises the expected value of future consumption which compensates for higher variance. Sibley (1975) and Miller (1974) extend this proof to a multi-period model with a finite  $T$ , and i.i.d. income shocks.

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<sup>17</sup>This result is a simple application of Stiglitz and Rothschild (1970)

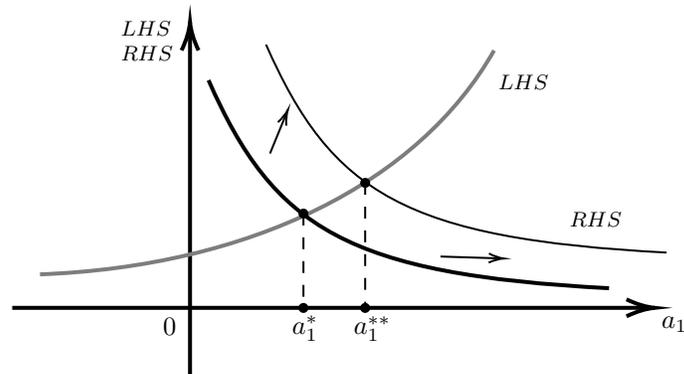


Figure 11.3: Left hand side (LHS) and right hand side (RHS) of the Euler equation (11.35).

**Notes:** It shows that when a mean-preserving spread in future income occurs, the RHS shifts outward and the optimal amount of saving in period zero  $a_1$  increases.

The convexity of the marginal utility corresponds precisely to the condition  $u''' > 0$ , or prudence. Prudence is a property of preferences, like risk-aversion: risk-aversion refers to the curvature of the utility function. Prudence refers to the curvature of the marginal utility function. Kimball (1990) defines the index of absolute prudence as  $-u'''(c)/u''(c)$  and relative prudence as  $-u'''(c)c/u''(c)$  in a conceptually similar way to the Arrow-Pratt index of absolute and relative risk-aversion. There exists also an interesting relationship between the two: any utility function in the decreasing absolute risk-aversion (DARA) class, which includes CRRA, displays prudence. To see this, let  $\alpha(c)$  be the coefficient of absolute risk-aversion, then

$$\alpha(c) = \frac{-u''(c)}{u'(c)} \Rightarrow \alpha'(c) = \frac{-u'''(c)u'(c) + [u''(c)]^2}{[u'(c)]^2}.$$

Since with DARA  $\alpha'(c) < 0$ , then we have that

$$-u'''(c)u'(c) + [u''(c)]^2 < 0 \Rightarrow u'''(c) > \frac{[u''(c)]^2}{u'(c)} > 0.$$

Intuitively, a rise in uncertainty reduces the certainty-equivalent income next period and, with DARA utility, it effectively increases the degree of risk-aversion of the agent, inducing them to save more.

### Role of prudence: an analytical expression

Blanchard and Mankiw (1988) derive an analytical expression that illustrates how risk affects optimal consumption and saving. Take a second-order approximation of  $u'(c_{t+1})$  around the value  $c_{t+1} = c_t$ :

$$u'(c_{t+1}) \simeq u'(c_t) + u''(c_t)(c_{t+1} - c_t) + \frac{1}{2}u'''(c_t)(c_{t+1} - c_t)^2.$$

Substitute this approximation into the right-hand side of the Euler equation  $u'(c_t) = \beta R \mathbb{E}_t u'(c_{t+1})$ , divide both sides by  $c_t^2$ , and use the approximation

$$(\beta R)^{-1} = \frac{1 + \rho}{1 + r} = \exp\left(\log \frac{1 + \rho}{1 + r}\right) \simeq 1 + \rho - r.$$

Rearranging terms, we obtain

$$\mathbb{E}_t \left( \frac{c_{t+1} - c_t}{c_t} \right) \simeq EIS(c_t) \cdot (r - \rho) + \frac{1}{2} P(c_t) \cdot \mathbb{E}_t \left[ \left( \frac{c_{t+1} - c_t}{c_t} \right)^2 \right] \quad (11.36)$$

where

$$EIS(c_t) \equiv -\frac{u'(c_t)}{c_t u''(c_t)}, \text{ and } P(c_t) = -\frac{u'''(c_t) c_t}{u''(c_t)}$$

are, respectively, the elasticity of intertemporal substitution and the coefficient of relative prudence. Equation (11.36) features the two determinants of expected consumption growth. The first term captures the standard intertemporal consumption smoothing motive, active as long as  $u$  is strictly concave. The second term contains the role of risk, captured by the expected variability of consumption growth (measured by the second uncentered moment): the stronger is relative prudence, the more consumption growth will respond to changes in risk. Higher uncertainty in future consumption growth, for given expected income, implies higher saving through the precautionary motive.

Similarly to the case where debt limits are binding, one can prove that even in absence of constraints, the consumption function is concave in wealth if the utility function belongs to the hyperbolic absolute risk aversion (HARA) class and displays positive third derivative.<sup>18</sup> First, recall that the envelope condition of the household problem yields

$$u'(c(a)) = V'(a) \quad (11.37)$$

where  $V$  denotes the value function and where, to ease notation, we have omitted the dependence on labor income. It can be proved that, under general conditions, the value function inherits the assumptions on  $u$ , i.e. it is increasing, concave and differentiable—see the discussions in Section 4.4. Differentiating both sides with respect to  $a$  gives

$$c'(a) = \frac{V''(a)}{u''(c(a))} > 0, \quad (11.38)$$

which implies that optimal consumption is strictly increasing in wealth. Differentiating one more time, we obtain

$$c''(a) = \frac{V'''(a) u''(c(a)) - V''(a) u'''(c(a)) c'(a)}{[u''(c(a))]^2}. \quad (11.39)$$

<sup>18</sup>A utility function belongs to the HARA class if its coefficient of absolute risk aversion is an hyperbola (or, equivalently, absolute risk tolerance is an affine function of wealth). Many common utility functions, such as quadratic, CRRA and CARA are special cases of HARA.

Using (11.37) and (11.38) into (11.39), we conclude that the consumption function is concave ( $c'' \leq 0$ ) whenever

$$\frac{V'''(a) V'(a)}{[V''(a)]^2} \geq \frac{u'''(c(a)) u'(c(a))}{[u''(c(a))]^2}. \quad (11.40)$$

This is the content of Lemma 2 in [Carroll and Kimball \(1996\)](#). If  $u$  belongs to the HARA class, then one can show that the right hand side of (11.40) is equal to a constant  $\kappa \geq 0$ , i.e. prudence is  $\kappa$  times higher than risk aversion.<sup>19</sup> Assuming HARA, [Carroll and Kimball \(1996\)](#) and [Jensen \(2018\)](#) also prove that, under a finite optimization horizon, the inequality in (11.40) is true and therefore the consumption function is concave. Strict concavity arises for  $\kappa > 0$ , which holds for the CRRA class, with the exception of  $\kappa = 1$  corresponding to CARA utility.<sup>20</sup>

Once again, the intuition is that the precautionary saving motive is declining in wealth. Consider the deviation in optimal consumption in the presence of uninsurable risk relative to the no-uncertainty case in which the consumption function is linear. For large enough wealth, precautionary saving approaches zero and the consumption function asymptotes the no uncertainty case. As wealth falls, precautionary saving keeps rising and consumption keeps falling, hence the concavity.

### Capital income uncertainty

What if the income uncertainty refers to capital income, i.e. to the rate of return on saving  $r$ , instead of labor income? Does an increase in uncertainty still induce more saving? Consider a problem analogous to the one analyzed earlier in this section. The household at date  $t = 0$  receives an income  $y_0$  which they can either consume or save, i.e.  $y_0 = c_0 + a_1$ . In the second period, consumption is simply  $c_1 = (1 + r) a_1$ , but  $r > -1$  is now stochastic. The Euler equation corresponding to this problem is

$$u'(y_0 - a_1) = \beta \mathbb{E}_0 [(1 + r) u'((1 + r) a_1)].$$

Note that the random interest rate is now inside the expectation. As before the left-hand side is increasing in  $a_1$  and the right-hand side decreasing in  $a_1$ , hence the solution is unique. What happens to optimal saving  $a_1$  after a mean-preserving spread in  $r$ ? The answer is not obvious ex-ante. On the one hand, the precautionary saving force would suggest that savings optimally expand. On the other hand, with higher savings, the household becomes even more exposed to risk, and thus reducing saving curtails risk. This latter force was not present when we analyzed labor income risk.

Under prudence, however, the right-hand side is still a convex function of  $r$  (it is the product of a linear and a convex function), and thus with this additional assumption

<sup>19</sup>Specifically, quadratic utility is the knife-edge case, corresponding to  $\kappa = 0$ , constant absolute risk aversion (CARA) corresponds to  $\kappa = 1$ , and constant relative risk aversion (CRRA) utility functions satisfy  $\kappa > 1$ .

<sup>20</sup>In fact, [Cantor \(1985\)](#) proved that the optimal solution to the consumption/saving problem of a CARA consumer facing uninsurable labor income risk (and no borrowing constraints) displays consumption which is linear in wealth. Compared to the no-uncertainty case, uninsurable risk only affects the intercept of the consumption function by lowering it. The slope (i.e., the constant MPC) is the same.

we can conclude that precautionary saving behavior emerges even in the face of higher volatility of returns.

### Buffer-stock saving

Consider a version of the intertemporal consumption problem where households have a finite horizon, utility is CRRA with curvature parameter  $\sigma > 0$ , log income follows a stochastic process which is the sum of a permanent shock  $y_t^P$  and a transitory shock  $u_t$  (as in Section 11.3.2, but in logs), both Normally distributed. In addition, there is positive probability that income will be zero at any  $t$  (capturing, e.g., an unemployment shock) and, as a result, the natural borrowing limit is zero. Finally, the discount rate  $\rho$  exceeds the interest rate  $r$ .<sup>21</sup>

Optimal consumption in this setting, called “buffer-stock model,” is fully characterized in a series of papers by [Carroll \(1992\)](#) and [Carroll \(2001\)](#). What makes this model particularly simple to analyze is that the individual state space can be reduced to only one state variable,  $x_t$ , the ratio of cash in hand  $Ra_t + y_t$  to permanent income  $y_t^P$ . Similarly to our derivation of equation (11.36), and focusing on isoelastic utility with curvature parameter  $\sigma$ , one can show that

$$\mathbb{E}_t \Delta \log c_{t+1} \simeq \frac{1}{\sigma} (r - \rho) + \frac{\sigma + 1}{2} \text{Var}_t (\Delta \log c_{t+1}), \quad (11.41)$$

where  $\text{var}_t$  denotes the conditional variance, and where  $\Delta \log c_t$  replaces the growth rate which appeared in equation (11.36).<sup>22</sup> As  $x_t \rightarrow 0$ , expected consumption growth approaches infinity because  $c_t$  approaches zero: when  $x_t = 0$  from the budget constraint  $-c_t = a_{t+1}/R \geq 0$ , and thus  $c_t = 0$ . As  $x_t \rightarrow \infty$ , uncertainty about future labor income becomes essentially irrelevant and expected consumption growth approaches  $\sigma^{-1} (r - \rho) < 0$ .

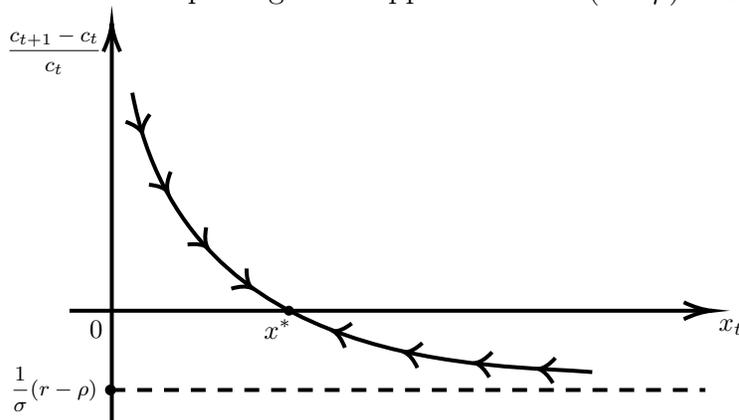


Figure 11.4: Consumption dynamics in the buffer-stock model as a function of cash-in-hand  $x$ .

**Notes:** There exists an optimal target level of cash in hand  $x$  toward which the household wants to move.

Figure 11.4 illustrates the relation between consumption growth and normalized wealth

<sup>21</sup>As we prove in Section 11.4,  $r < \rho$  is an equilibrium outcome in an economy populated by a continuum of this type of consumers.

<sup>22</sup>For a full derivation, see for example [Jappelli and Pistaferri \(2017\)](#), chapter 6.

$x$ . The figure highlights that there exists a level of wealth  $x^*$  that is dynamically stable. If  $x_t > x^*$  the negative intertemporal saving motive outweighs the precautionary saving motive. Consumption growth is negative, and wealth converges back toward  $x^*$ . If  $x_t < x^*$ , the precautionary motive dominates, consumption growth is positive, and wealth rises toward  $x^*$ . In sum, households have a *target level of assets*  $x^*$  which is above the level of the no uncertainty case, i.e., they hold additional wealth as a buffer stock. Labor income shocks take households periodically above or below this target level, and households adjust their optimal consumption in order to quickly move back to  $x^*$ .

### Excess sensitivity and liquidity constraints

According to the permanent income hypothesis, consumption growth between  $t - 1$  and  $t$  should only depend on the innovation in income accruing over the same period. Any past income change should not affect current consumption growth. Empirical tests of this prediction routinely lead to a rejection (e.g., [Flavin, 1981](#)) of this hypothesis and to evidence of so-called “excess sensitivity.” [Deaton \(1992\)](#) and [Jappelli and Pistaferri \(2017\)](#) discuss various econometric difficulties with this type of tests. [Campbell and Mankiw \(1989\)](#) also found evidence of excess sensitivity on aggregate time series data, and interpreted the size of the coefficient on past income as the share of “hand-to-mouth” consumers, i.e. households that are either myopic or subject to liquidity constraints.

[Zeldes \(1989\)](#) links explicitly excess sensitivity tests to the presence of credit constraints. If we generalize equation (11.41) to the case where households face a borrowing limit, we obtain the following Euler equation for consumption:

$$\Delta \log c_{t+1} = \frac{1}{\sigma}(r - \delta) + \frac{\sigma + 1}{2} \text{Var}_t(\Delta \log c_{t+1}) + \frac{1}{\sigma} \log(1 + \lambda_t) + \varepsilon_{t+1}$$

where  $\lambda_t$  is the multiplier on the borrowing limit, and  $\varepsilon_{t+1}$  is a forecasting error orthogonal to consumption growth. The coefficient on  $\lambda_t$  is positive because when the marginal value of wealth is high, households save more (and consume less) at  $t$ , which pushes up consumption growth between  $t$  and  $t + 1$ . Note that the multiplier is inherently unobservable, so it acts as an omitted variable.

Zeldes splits his empirical sample of PSID households into low-wealth (likely constrained) and high-wealth (likely unconstrained) households, and adds lagged income as a regressor to that equation to proxy for the multiplier. He argues that, if liquidity constraints bind, one would estimate a significant negative coefficient on lagged income for the low-wealth sample. The reason is that a high income realization relaxes the constraint and lowers the value of the multiplier  $\lambda_t$ . Zeldes’ empirical estimates are consistent with the view that liquidity constraints bind for a subgroup of the population, but not for all.

### Comparative statics on the consumption function

Figure 11.5 plots the optimal consumption function obtained by solving numerically the income fluctuation problem (11.31). In particular, we have assumed that  $u$  is CRRA with risk aversion equal to 2,  $\beta = 0.95$ ,  $R = 1.02$ , the borrowing limit is zero, and income follows

a two-state Markov chain with values  $\{0.5, 1.5\}$ , and probability that each income state persists into next period equal to 0.8.

The six panels illustrate how the solution to the household problem changes as we change one parameter at the time. Panel (A) shows that households with a low income realization consume less on average for two reasons. First, mechanically, they have less resources. Second, when they are not constrained they are more concerned about hitting the constraint in the future and thus have a stronger precautionary motive. Note that, as wealth grows the households becomes a permanent-income consumer, and the consumption gap converges to the expected difference in permanent labor income. In particular, for wealth large enough the slope of the consumption function is the same. Turning to panel (B), more impatient households (low discounting) consume more and, as long as they are unconstrained, have a higher MPC, even for high values of wealth. Recall the expression for the MPC under full insurance in equation (11.15). Panel (C) shows that, since  $\beta R < 1$  households who are more willing to substitute intertemporally consume more. In addition, as seen from (11.15), their MPC is larger even for high levels of wealth. Panel (D) illustrates that households who face a higher rate of return on saving optimally accumulate more assets and consume less for a given level of wealth (but they will be richer on average). Panel (E) shows that a mean-preserving spread in income risk lowers consumption because it expands precautionary saving. For the same reason, the credit constraint binds for a smaller wealth range. Finally, panel (F) depicts optimal consumption under two values for the credit limit. Households who face a loose constraint save less for precautionary reasons. This result indicates the existence of substitutability between precautionary saving and ability to borrow to smooth consumption when needed.

### 11.3.5 Bounds on wealth accumulation

As discussed, there are two forces that determine saving rates in the stochastic income fluctuation problem: intertemporal and precautionary motive. The latter is always positive. Whereas the first is positive or negative depends on whether  $\beta R$  is above or below one. It is therefore intuitive that unless  $\beta R$  is small enough, households will tend to keep accumulating savings, and an upper bound on wealth might not exist. This result represents a threat to existence of equilibrium once we combine a continuum of households and let them trade in financial markets: if assets can grow without limit, the state space is not compact and an equilibrium might not exist.<sup>23</sup>

It turns out that a sufficient condition for wealth accumulation to be bounded in the stochastic version of the income fluctuation problem is  $\beta R < 1$  (Schechtman and Escudero, 1977; Clarida, 1987; Huggett, 1997).<sup>24</sup> We now provide an informal version of the proof of this result based on Schechtman and Escudero (1977): besides  $\beta R < 1$ , the proof assumes i.i.d. income shocks. We use a recursive formulation where ‘prime’ denotes next-period variables. To avoid confusion, here we use the special notation where  $u_c(\cdot)$  is the first derivative of the utility function (the marginal utility of consumption) and  $u_{cc}$  is the second derivative of

<sup>23</sup>A lower bound for the asset space is always guaranteed by the ad-hoc or natural borrowing limit.

<sup>24</sup>The counterpart sufficient condition in the deterministic version of the income fluctuation problem is  $\beta R \leq 1$ . It is a weaker condition because the precautionary saving motive is absent. See Ljungqvist and Sargent (2018) for a thorough discussion of both the deterministic and the stochastic case.

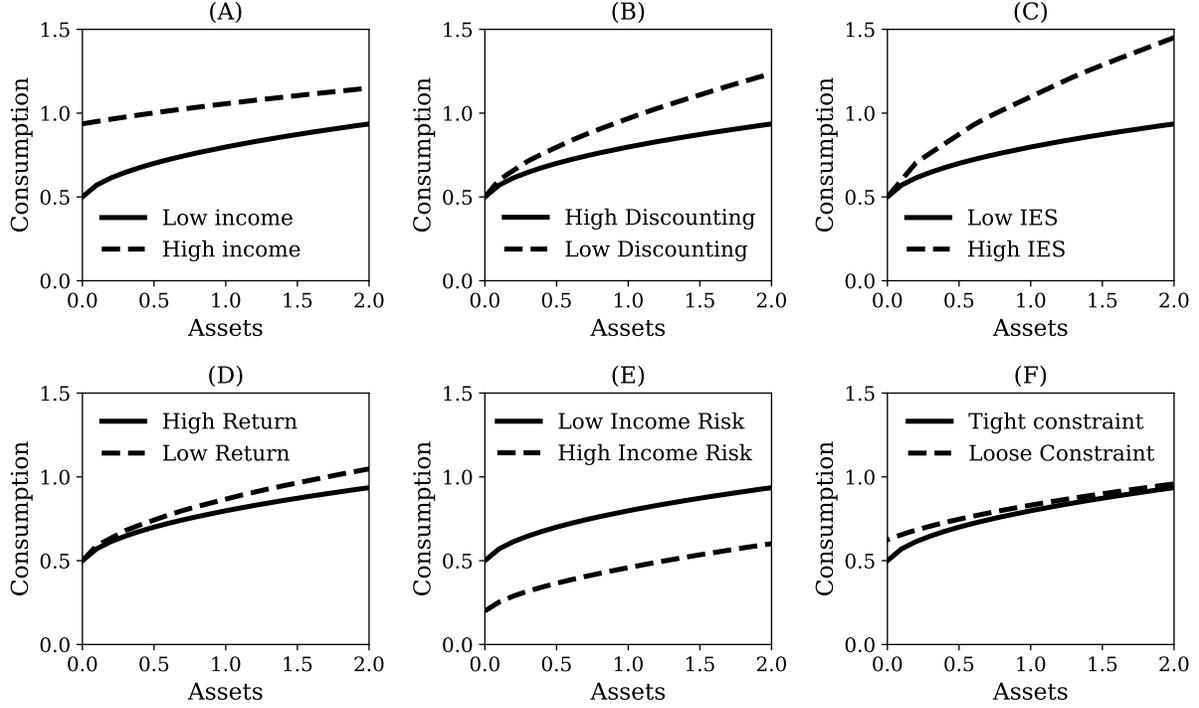


Figure 11.5: Comparative statics of the optimal consumption function with respect to various parameters in the income fluctuations problem.

the utility function. Let  $x = Ra + y$  be cash in hand, and  $c(x)$  the consumption decision rule—a function of the individual state variable  $x$ .  $c_x(\cdot)$  represents the first derivative of this function. From the Euler Equation:

$$u_c(c(x)) = \beta R \mathbb{E}[u_c(c(x'))], \quad (11.42)$$

where  $x' = Ra'(x) + y'$  is cash in hand next period, given that today's cash in hand is  $x$  and that next period income realization is  $y'$ . Let  $\bar{x}' = Ra'(x) + \bar{y}$  be the cash in hand associated to the maximum realization of income  $\bar{y}$  next period (hence maximum cash in hand next period), given today's cash in hand  $x$ . We can write:

$$u_c(c(x)) = \beta R \mathbb{E}[u_c(c(x'))] = \beta R \frac{\mathbb{E}[u_c(c(x'))]}{u_c(c(\bar{x}'))} u_c(c(\bar{x}')). \quad (11.43)$$

Suppose we can show that

$$\lim_{x \rightarrow \infty} \frac{\mathbb{E}[u_c(c(x'))]}{u_c(c(\bar{x}'))} = 1. \quad (11.44)$$

Then, since  $\beta R < 1$  by assumption, for  $x$  large enough the Euler equation (11.43) yields

$$u_c(c(x)) = \beta R u_c(c(\bar{x}')) < u_c(c(\bar{x}')).$$

Concavity of  $u$  and monotonicity of  $c$  with respect to  $x$ , proved in (11.38) imply that  $\bar{x}'(x) < x$ . This would conclude the proof, because we would have demonstrated that cash in hand does not increase forever: when  $x$  is large enough,  $x' < x$  for sure.

We need to establish conditions under which the limit in (11.44) holds. Consider the marginal utility function  $u_c(c(x'))$  and compute a first-order Taylor approximation around  $x' = \bar{x}'$ :

$$u_c(c(x')) \simeq u_c(c(\bar{x}')) + u_{cc}(c(\bar{x}')) c_x(\bar{x}') (x' - \bar{x}').$$

Taking expectations of both sides

$$\begin{aligned} \mathbb{E}[u_c(c(x'))] &\simeq u_c(c(\bar{x}')) - u_{cc}(c(\bar{x}')) \mathbb{E}[\bar{x}' - x'] c_x(\bar{x}') \\ &= u_c(c(\bar{x}')) - u_{cc}(c(\bar{x}')) \mathbb{E}[\bar{y} - y'] c_x(\bar{x}'). \end{aligned} \quad (11.45)$$

In the first line we used that  $\bar{x}'$  is deterministic since it is implied by the specific income realization  $\bar{y}$ . In the second line we used the fact that  $x' \equiv Ra' + y'$  which implies  $\bar{x}' - x' \equiv \bar{y} - y'$ . Dividing equation (11.45) by  $u_c(c(\bar{x}'))$  we obtain:

$$\frac{\mathbb{E}[u_c(c(x'))]}{u_c(c(\bar{x}'))} \simeq 1 + \alpha(c(\bar{x}')) [\bar{y} - \mathbb{E}(y')] c_x(\bar{x}'),$$

where  $\alpha(c(\bar{x}'))$  is the coefficient of absolute risk aversion evaluated at consumption level  $c(\bar{x}')$ . Since both  $\bar{y} - \mathbb{E}(y')$  and  $c_x(\bar{x}')$  are positive and finite (recall that the  $c$  function is concave), a sufficient condition for the limit in (11.44) to hold is that

$$\lim_{x \rightarrow \infty} \alpha(c(x)) = 0. \quad (11.46)$$

Condition (11.46) requires that absolute risk aversion decreases with wealth, i.e. that  $u$  belongs to the DARA class. DARA means that the household is less and less concerned about income uncertainty as they get rich because they are less risk averse, so they will consume more and accumulate less precautionary wealth. As wealth increases, the intertemporal dissaving motive (present because of  $\beta R < 1$ ) eventually overcomes the precautionary saving motive, and wealth will decrease.<sup>25</sup>

To conclude, under DARA the condition  $\beta R < 1$  is sufficient to guarantee a bounded asset space in the household problem. In the next section, we'll prove that this inequality is also a feature of the stationary competitive equilibrium in an economy with a continuum of households.

## 11.4 Heterogeneous-agent incomplete-market models

Now that we have analyzed in some depth the income fluctuation problem, we want to characterize the equilibrium of economies populated by a continuum of households subject to uninsurable income shocks who can only trade a risk-free bond. This class of heterogeneous-agent incomplete-market models has become a workhorse of modern macroeconomics. These models feature an endogenous distribution of consumption and wealth across households. In sharp contrast with its complete-market counterpart, this framework features partial consumption insurance, and mobility within the consumption distribution. In addition, its equilibrium allocations are not socially efficient, which makes policy analysis interesting.

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<sup>25</sup>Huggett (1993) generalizes this proof to a 2-state Markov chain for the income process, under CRRA utility.

A variety of important questions have been addressed in this framework, for example: what is the fraction of wealth accumulated because of the precautionary motive (Aiyagari, 1994)? How much of the observed wealth inequality can one explain through income inequality (Castañeda, Díaz-Giménez, and Ríos-Rull, 2003)? What are the optimal levels of taxation and public debt (Aiyagari and McGrattan, 1998; Domeij and Floden, 2006; Heathcote, 2005)? How does heterogeneity change the transmission of aggregate shocks compared to the representative agent model (Krusell and Smith, 1998; Kaplan and Violante, 2018)? Can market incompleteness help in generating a large equity premium (Storesletten, Telmer, and Yaron, 2001, Krueger and Lustig, 2010)? Are welfare costs of business cycles higher with incomplete markets (Krusell, Mukoyama, Şahin, and Smith Jr, 2009)?<sup>26</sup>

In what follows, we describe two versions of the model, an endowment economy and a production economy, and discuss existence and uniqueness of the stationary equilibrium.

### 11.4.1 An endowment economy

Consider the endowment economy introduced in Section 7.6. Time is discrete and indexed by  $t$ . There is no aggregate uncertainty. The economy is populated with a continuum of measure one of infinitely lived, ex-ante identical households. Households have time-separable preferences over streams of consumption (the final numeraire good)

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

where the period utility function  $u$  satisfies  $u' > 0$ ,  $u'' < 0$ , and the discount factor  $\beta \in (0, 1)$ . The expectation is over future sequences of idiosyncratic shocks, conditional on the information set at time  $t = 0$ . Each household faces stochastic fluctuations in their endowment of the final good  $y_t \in Y$ , a finite set. This stochastic process follows a first-order ergodic  $N$ -state Markov chain with transition probabilities  $\pi(y', y) = \Pr(y_{t+1} = y' \mid y_t = y)$ . Since shocks are i.i.d across consumers, a law of large numbers holds:  $\pi(y', y)$  is also the fraction of agents in the population subject to this particular transition from  $y$  to  $y'$  between time  $t$  and  $t + 1$  (Uhlig, 1996). Let  $\Pi^*(y)$  be the a unique invariant (i.e. limiting) distribution of endowments. The household budget constraint at time  $t$  is

$$c_t + a_{t+1} = Ra_t + y_t.$$

Wealth  $a_t$  takes the form of a one-period non-state-contingent asset, with return  $R \equiv 1 + r$  independently of the individual realization  $y_t$ . At every  $t$ , agents face the ad hoc borrowing limit

$$a_{t+1} \geq -\underline{a}$$

where  $\underline{a}$  is a parameter which we assume is more stringent than the natural debt limit. We begin by positing that this asset is in zero aggregate net supply, as in Huggett (1993), and traded among households in a competitive financial market.

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<sup>26</sup>For surveys of applications of this framework, see Heathcote, Storesletten, and Violante (2009); Quadrini and Ríos-Rull (2015); De Nardi and Fella (2017); Benhabib, Bisin, and Luo (2019); Kaplan and Violante (2022).

## Recursive formulation

We now restate the economy in recursive form. The dynamic programming version of the household problem described above is

$$V(a, y) = \max_{c, a'} \left\{ u(c) + \beta \sum_{y' \in Y} \pi(y', y) V(a', y') \right\} \quad (11.47)$$

subject to

$$c + a' = Ra + y$$

and

$$a' \geq -\underline{a},$$

where  $V$  denotes the value function, and the pair  $(a, y)$  is the individual state vector.

Let  $\lambda$  be the probability distribution of agents over states. Let  $\bar{a}$  be the maximum asset holding in the economy, and for now assume that such upper bound exists. Let  $A \equiv [-\underline{a}, \bar{a}]$  be the asset space. Let the state space  $S$  be the Cartesian product  $A \times Y$ . Let the  $\sigma$ -algebra  $\Sigma_s$  be defined as  $B_A \otimes P(Y)$  where  $B_A$  is the Borel  $\sigma$ -algebra on  $A$  and  $P(Y)$  is the power set of  $Y$ . The space  $(S, \Sigma_s)$  is a measurable space. Let  $\mathcal{S} = (\mathcal{A} \times \mathcal{Y})$  be the typical set of  $\Sigma_s$ . For any element of the  $\sigma$ -algebra  $\mathcal{S} \in \Sigma_s$ ,  $\lambda(\mathcal{S})$  is the measure of agents in the set  $\mathcal{S}$ .

How can we characterize the way individuals transit across states over time? We need a transition function. Let  $Q((a, y), \mathcal{A} \times \mathcal{Y})$  be the (conditional) probability that an individual with current state  $(a, y)$  transits to the set  $\mathcal{A} \times \mathcal{Y}$  next period. Formally,  $Q : S \times \Sigma_s \rightarrow [0, 1]$ , and

$$Q((a, y), \mathcal{A} \times \mathcal{Y}) = \mathbb{I}_{\{a'(a, y) \in \mathcal{A}\}} \sum_{y' \in \mathcal{Y}} \pi(y', y), \quad (11.48)$$

where  $\mathbb{I}_{\{\cdot\}}$  is the indicator function, and  $a'(a, y)$  is the saving policy. This transition function can be used to construct a sequence of distributions as

$$\lambda_{n+1}(\mathcal{A} \times \mathcal{Y}) = \int_{A \times Y} Q((a, y), \mathcal{A} \times \mathcal{Y}) d\lambda_n. \quad (11.49)$$

We are now ready to proceed to the definition of a stationary, or steady-state, equilibrium.

## Stationary equilibrium

A recursive stationary competitive equilibrium is: (a) value function  $V : S \rightarrow \mathbb{R}$ , (b) policy functions  $a' : S \rightarrow \mathbb{R}$ , and  $c : S \rightarrow \mathbb{R}_+$ , (c) an interest rate  $r^*$ , and (d) a stationary measure  $\lambda^*$  such that:

- Given  $r$ , the policy functions  $a'$  and  $c$  solve the household's problem (11.47) and  $V$  is the associated value function
- The asset market clears:  $\int_{A \times Y} a'(a, y) d\lambda^* = 0$  at the interest rate  $r^*$
- The goods market clears:  $\int_{A \times Y} c(a, y) d\lambda^* = \sum_{i=1}^N y_i \Pi^*(y_i)$

- For all  $(\mathcal{A} \times \mathcal{Y}) \in \Sigma_s$ , the invariant probability measure  $\lambda^*$  is the fixed point of (11.49) and satisfies

$$\lambda^*(\mathcal{A} \times \mathcal{Y}) = \int_{\mathcal{A} \times \mathcal{Y}} Q((a, y), \mathcal{A} \times \mathcal{Y}) d\lambda^*,$$

where  $Q$  is the transition function defined in (11.48).

In the stationary equilibrium, households optimize, markets clear, and the distribution of agents across states is invariant, i.e. this probability measure will reproduce itself permanently. Households, however, will exchange places and move upward and downward within this joint distribution of income and wealth, since they face different sequences of endowment shocks.

What guarantees that a steady-state exists and is unique? We start from existence.

**Existence.** Let us express the excess aggregate demand function for assets in financial markets as

$$A(r) = \int_{\mathcal{A} \times \mathcal{Y}} a'(a, y; r) d\lambda_r^*,$$

where we made explicit the dependence of the policy function and the stationary distribution on the interest rate  $r$ ; think of  $r$  as a parameter for now. One of the conditions that yield existence is that  $A(r)$  is continuous in  $r$ .  $A(r)$  is continuous if both  $a'$  and  $\lambda_r^*$  are, in turn, continuous in  $r$ .

The theorem of the maximum implies that if  $u$  is continuous,  $u' > 0$  and  $u'' < 0$ , the solution to the household problem is unique and the policy function  $a'(a, \varepsilon; r)$  is continuous in  $r$ .

Stokey, Lucas and Prescott (Theorems 12.12 and 12.13) lay out conditions under which  $\lambda_r^*$  is continuous in  $r$ . Essentially, the stationary distribution must exist and be unique. We refer the interested reader to Stokey, Lucas and Prescott for the exact conditions needed and to Huggett (1993) and Aiyagari (1994) for proofs that these assumptions are met in this environment. Most of these proofs are straightforward. We highlight two conditions that deserve more discussion, one for existence and one for uniqueness.

Existence of an invariant distribution requires compactness of the state space, i.e., a finite upper bound for wealth  $a$ . As explained,  $\beta R < 1$  is a sufficient condition to obtain a finite upper bound for  $a$ , but  $R$  is endogenous so this restriction cannot be assumed. It must hold in equilibrium, and later in this section we show that it is true.

Uniqueness of the invariant distribution is guaranteed if the transition function  $Q((a, y), \mathcal{A} \times \mathcal{Y})$  satisfies, for any given  $r$ , the “monotone mixing condition” (and a few regularity conditions). This condition states that there is a positive probability that a household moves from the bottom to the top of the asset space (and vice versa) in finite time. Namely, the economy must have enough upward and downward mobility within the income and wealth distribution. To see why this condition is satisfied in our model, suppose the household starts from  $(\bar{a}, y_{\max})$  and receives a long stream of the worst realization of the shock  $y_{\min}$ . If  $y$  follows a stationary (i.e., mean reverting) process, the household will keep decumulating wealth to smooth consumption until reaching some neighborhood of the lower bound. Symmetrically, suppose the household starts with  $(-\underline{a}, y_{\min})$  and receives a long stream of the best shock

$y_{\max}$ . Knowing that sooner or later the shock will revert toward its mean, they will keep accumulating wealth until they reach some neighborhood of the upper bound.

Having proved that the  $A(r)$  function is continuous, we need to argue that it crosses zero at least once at some finite value for  $r$ . For  $\beta(1+r) = 1$ , we know from Section 11.3.5 that households will keep accumulating wealth without bound, so  $A\left(\frac{1}{\beta} - 1\right) = +\infty$ . For low enough (possibly negative) values of  $r$ , every household would want to borrow. For example, it is clear that if  $r = -1$ , it is optimal to borrow up to the limit since one would never have to repay, and thus  $A(-1) = -\underline{a}$ . Since  $A(r)$  is continuous, it will cross zero at least once and an equilibrium exists. This logic is represented graphically in Figure 11.6. Note that the equilibrium interest rate  $r^*$  will always lie below its complete market counterpart  $1/\beta - 1$ . As a result, the condition  $\beta R^* < 1$  holds in equilibrium, which puts a limit to wealth accumulation and confirms that the asset space is bounded above.

### A solution to the risk-free rate puzzle

The observation that the equilibrium interest rate is lower than under full insurance is important for asset pricing, as emphasized by [Huggett \(1993\)](#). The representative agent model can only generate a high equity premium for very large levels of risk aversion ([Mehra and Prescott, 1985](#)) which, under CRRA utility, imply very low values for the intertemporal elasticity of substitution. In this situation, households have a very sharp desire to smooth consumption. When income has positive growth on average, as in the data, consumption smoothing dictates a strong desire to borrow against future income which pushes up the real rate well above observed values.

The cost of solving the equity premium puzzle with complete markets is, therefore, the “risk-free rate puzzle”: the return on a risk-free bond is too high compared to the historical data. [Huggett \(1993\)](#) showed that the presence of uninsurable idiosyncratic income risk can help solving the puzzle. As soon as one deviates from complete markets, the precautionary saving motive sets in (and a strong one if risk aversion is high) and, with it, a desire to save that pulls down the equilibrium interest rate on safe assets and better aligns it to the data.

**Uniqueness.** Having proved that the equilibrium exists, we ask: is the equilibrium unique? Uniqueness is guaranteed if the function  $A(r)$  is monotonically increasing in  $r$ . This property is hard to prove in general because changes in  $r$  have both income and substitution effects on savings: the relative dominance between the two could switch at a certain level of assets, so  $a'(a, \varepsilon; r)$  may not be monotone in  $r$ . [Achdou, Han, Lasry, Lions, and Moll \(2022\)](#) state a sufficient condition for CRRA utility in a continuous time version of this model. Let  $\sigma$  be the coefficient of risk aversion. Recall that the smaller is  $\sigma$  the larger is the income effect and a higher  $r$  increases savings. Uniqueness is obtained if  $\sigma < 1$ .

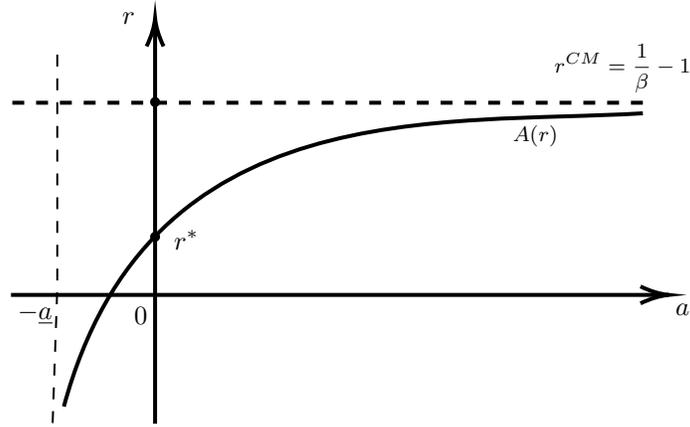


Figure 11.6: Equilibrium of the endowment economy where assets are in zero net supply.

**Notes:**  $A(r)$  denotes aggregate asset holdings of the household sector as a function of the interest rate in the incomplete-market model.  $r^{CM}$  denotes the complete markets equilibrium real rate, and also the infinitely elastic demand for assets in the complete market model.  $r^*$  is the equilibrium real rate in the model with incomplete markets.

### Assets in positive supply

We conclude this section by extending this logic to an endowment economy where assets are not in zero net supply, but are one-period real bonds (or real balances of fiat money) issued by the government. This is the economy studied by [Bewley \(1980, 1983\)](#). Let  $B$  be this amount, fixed in steady-state. The government budget constraint is

$$rB = T, \tag{11.50}$$

where  $T$  is the lump-sum tax (if  $r > 0$ ) levied on households to finance interest payments. This is a new equilibrium condition determining  $T$ . The household budget constraint in (11.47) is modified as

$$c + a' = Ra + y - T,$$

and the equilibrium condition in the asset market, represented graphically in Figure 11.7, becomes

$$\int_{A \times Y} a'(a, y; r) d\lambda_r^* = B.$$

The analysis of equilibrium is virtually the same as in the previous section, with the only modification that we have an additional endogenous variable,  $T$ , and an additional equation, (11.50).

We conclude by noting that, in this economy, the government would have an incentive to provide more liquidity to households by increasing the level of debt  $B$ . Returning to Figure 11.7, as  $B$  increases, the economy moves closer to the full insurance outcome. In the economy with production that we analyze next, however, expanding government debt also entails a cost, because it crowds out capital and production. [Aiyagari and McGrattan \(1998\)](#) analyze this trade-off in the determination of the optimal quantity of debt.

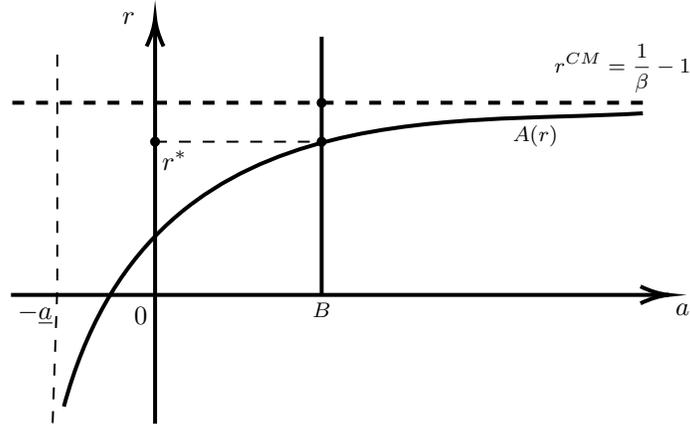


Figure 11.7: Equilibrium of the endowment economy where assets are in fixed positive supply.

**Notes:**  $A(r)$  denotes aggregate asset holdings of the household sector as a function of the interest rate in the incomplete-market model.  $r^{CM}$  denotes the complete markets equilibrium real rate, and  $r^*$  the equilibrium real rate in the model with incomplete markets.

### 11.4.2 A production economy

We now follow [Aiyagari \(1994\)](#) and introduce production in this economy. We re-interpret  $y$  as efficiency units of labor (i.e., individual labor productivity) and assume that households supply labor inelastically, with each individual time endowment normalized to 1. The asset  $a$  represents financial claims to physical capital. Households supply labor and capital to a representative firm. The firm produces the final good with CRS production function  $F(K, L)$ , which uses capital and efficiency units of labor as inputs and satisfies  $F_K > 0, F_L > 0, F_{KK} < 0, F_{LL} < 0$  as well as standard Inada conditions,  $\lim_{K \rightarrow \infty} F_K = 0$  and  $\lim_{K \rightarrow 0} F_K = +\infty$ . Physical capital depreciates geometrically at rate  $\delta \in (0, 1)$ . Firms act competitively by maximizing profits taking prices as given. The final numeraire good can be used for consumption and investment, and is traded in a competitive good market. The labor market and capital market are also competitive and clear, respectively, at the wage rate  $w$  (per efficiency unit) and interest rate  $r$ .

The household budget constraint in problem (11.47) becomes

$$c + a' = Ra + wy$$

and everything else in the household problem is unchanged. We now state the new definition of equilibrium.

A recursive stationary competitive equilibrium is: (a) value function  $V : S \rightarrow \mathbb{R}$ , (b) policy functions  $a' : S \rightarrow \mathbb{R}$ , and  $c : S \rightarrow \mathbb{R}_+$ , (c) firm choices  $L$  and  $K$ , (d) prices  $r^*$  and  $w^*$ , and (e) a stationary measure  $\lambda^*$  such that:

- Given  $(r, w)$ , the policy functions  $a'$  and  $c$  solve the household's problem (11.47) and  $V$  is the associated value function
- Given  $(r, w)$ , the firm chooses optimally its capital stock  $K$  and its labor input  $L$ , i.e.,  $r + \delta = F_K(K, L)$  and  $w = F_L(K, L)$

- The labor market clears:  $L = \sum_{i=1}^N y_i \Pi^*(y_i)$  at the wage  $w^*$
- The asset market clears:  $\int_{\mathcal{A} \times \mathcal{Y}} a'(a, y) d\lambda^* = K$  at the interest rate  $r^*$
- The goods market clears:  $\int_{\mathcal{A} \times \mathcal{Y}} c(a, y) d\lambda^* + \delta K = F(K, L)$
- For all  $(\mathcal{A} \times \mathcal{Y}) \in \Sigma_s$ , the invariant probability measure  $\lambda^*$  satisfies

$$\lambda^*(\mathcal{A} \times \mathcal{Y}) = \int_{\mathcal{A} \times \mathcal{Y}} Q((a, y), \mathcal{A} \times \mathcal{Y}) d\lambda^*,$$

Production changes the shape of the aggregate demand for capital from firms. To characterize it, we need to consider the firm's problem. From the optimal choice of the firm, we obtain  $K(r)$  implicitly from  $F_K(K, L) = r + \delta$ . It is immediate to see that for  $r = -\delta$ , then  $K \rightarrow +\infty$ , while for  $r \rightarrow +\infty$ ,  $K \rightarrow 0$ , given our assumptions on  $F$ , in particular the Inada conditions. Thus, firms' demand for capital is a continuous, strictly decreasing function of the interest rate  $r$ .<sup>27</sup> Figure 11.8 illustrates the equilibrium in this model. Existence follows from the same conditions discussed above.

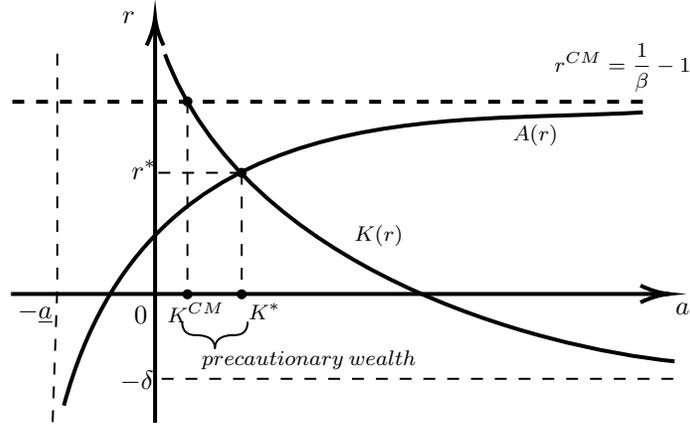


Figure 11.8: Equilibrium of the production economy.

**Notes:**  $A(r)$  denotes aggregate asset holdings of the household sector as a function of the interest rate in the incomplete-market model.  $r^{CM}$  and  $K^{CM}$  denote, respectively, the complete markets equilibrium real rate and capital stock.  $r^*$  and  $K^*$  are their incomplete markets counterparts. The difference between  $K^*$  and  $K^{CM}$  is the equilibrium amount of precautionary wealth.

### Quantitative analysis of the model.

Aiyagari (1994) calibrated the model to the U.S. economy. One of the key ingredients of the model, the stochastic process of earnings shocks (in particular, their persistence and volatility), is estimated from longitudinal micro data on individual labor income. Aiyagari reached three important conclusions from his quantitative analysis, all results that have shaped the literature for decades.

<sup>27</sup>For example, if  $F(K, L) = K^\alpha L^{1-\alpha}$ , then  $K(r) = \left(\frac{\alpha L}{\delta+r}\right)^{\frac{1}{1-\alpha}}$ .

First, he argued that this incomplete market structure is quite effective at insuring income risk. By saving, dissaving, and borrowing (i.e., through self-insurance), an individual can cut consumption variability by about half compared with autarky. Second, he pointed out that one can use the model to quantify the amount of aggregate wealth held by households because of precautionary reasons. The capital stock in complete markets is the value  $K^{CM}$  where the aggregate demand for capital by firms crosses the infinitely elastic supply of capital of the representative consumer at  $r^{CM} = 1/\beta - 1$ . The difference between  $K^*$  and  $K^{CM}$  is therefore the additional precautionary stock of capital. He concluded that, in his baseline calibration, the precautionary motive augments the aggregate saving rate by 3 percentage points, but calibrations featuring higher risk aversion or more income volatility can raise the aggregate saving rate by 10 percentage points or more. Third, he observed that the model can generate an equilibrium wealth distribution that is more positively skewed (mean > median) and more dispersed than the income distribution, as in the data. Compared to its empirical counterpart, though, the model's wealth distribution features too much wealth accumulation at the bottom and too little wealth concentration at the top. Chapter 21 describes the progress made in the literature on wealth inequality since Aiyagari's observation.

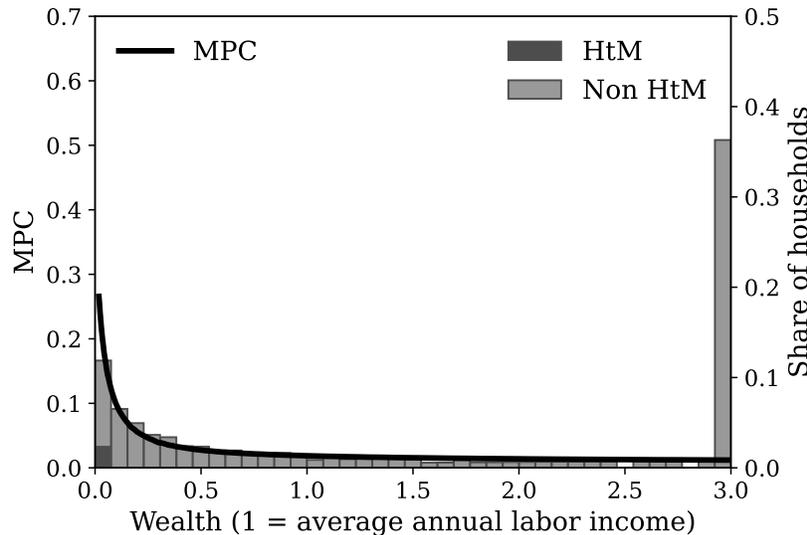


Figure 11.9: MPC, i.e. the slope of the consumption decision rule (curve), as a function of wealth jointly with the distribution of wealth (bars).

**Notes:** The dark bar near zero represents the share of constrained (or hand-to-mouth, HtM) households in the calibrated model. This figure is reproduced from [Kaplan and Violante \(2022\)](#).

### The marginal propensity to consume: models vs data

In Section 11.3.1 we showed that, under the permanent income hypothesis (PIH), the marginal propensity to consume (MPC) out of wealth and unanticipated transitory

changes in income equals  $1 - R^{-1}(\beta R)^{\frac{1}{\sigma}}$ . In the equilibrium of a representative agent economy, where  $\beta R^* = 1$ , the MPC therefore roughly equals the real interest rate, or around 1% on a quarterly basis. In addition, according to the PIH, the MPC out of anticipated income changes is zero.

There is a vast body of work that estimates the MPC out of (more or less anticipated) transitory changes in income using various approaches. Some of the most convincing evidence comes from the observation that when households receive fiscal stimulus payments from the government (like in the last three recessions, for example), they spend on average around 15-30% of this transfer on nondurable goods in the first quarter after receipt (e.g., [Parker, Souleles, Johnson, and McClelland, 2013](#)). These estimates indicate that, empirically, the MPC might be up to 20 times as large in the data than what implied by the naive PIH.

The heterogeneous-agent incomplete-market model we analyzed has the potential to match the data better. As discussed in [Section 11.3.4](#), for low enough wealth levels the slope of the consumption function can be much steeper than  $r$ . The key question is: in the equilibrium of a plausibly calibrated model, what is the share of agents with wealth holdings in that range? The answer is not many for two reasons. First, in order for the model to match the aggregate wealth level observed in the U.S., the model discount factor has to be high. Second, precisely because being on a constraint induces excessive consumption fluctuations that are costly in terms of welfare, optimization leads households to save and keep away from their credit limit. Overall, the quarterly MPC can be boosted up to 5% or so, but no more. [Figure 11.9](#), reproduced from [Kaplan and Violante \(2022\)](#) illustrates this result.

The literature has studied how to modify the baseline model to generate levels of average MPC more in line with the data. We refer to [Kaplan and Violante \(2022\)](#) for a comprehensive survey, and here only discuss two of them. First, one can introduce heterogeneity in discount factors across consumers. Households with low discount factors display steeper consumption functions (recall our comparative statics of [Section 11.3.4](#)). In addition, because of impatience, they tend to dissave and thus hold small amounts of wealth. With enough households of this type, the average MPC of the economy can be large. Second, one can take the view that the relevant notion of wealth for short-term consumption smoothing is *liquid wealth*, e.g., cash and bank deposits but not housing or retirement accounts. Empirically, liquid wealth is only a small fraction of total wealth. A model with two types of assets, a liquid one and an illiquid one which can only be accessed by paying a transaction cost, is able to yield averages MPC in line with the data. Interestingly, this type of models features *wealthy hand-to-mouth* consumers, i.e. consumers who do have substantial wealth locked in high-return illiquid assets, but hold small amounts of liquid wealth and thus are highly responsive to transitory income changes ([Kaplan and Violante, 2014](#); [Kaplan, Violante, and Weidner, 2014](#)).