Chapter 17

Money

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17.1 Introduction

So far, this textbook has exclusively discussed *real* variables. In the models covered, money is not present, and "dollars" have no special meaning. In microeconomics, on which the macroeconomic models here are based, the label money is sometimes used but merely to denote a numéraire. That is, the only prices that matter are relative prices between goods and services traded. In macroeconomics more broadly, however, money often takes center stage. First, people are concerned with *inflation* and the notion that it may erode the purchasing power of income. Second, one of the main macroeconomic policy tools available to governments is *monetary policy*, conducted by *central banks* by controlling the stock (supply) of money or the nominal interest rate, i.e., the rate of exchange between dollars at different points in time (overnight, or from year to year). Historically, monetary policy primarily meant money creation as a source of revenue (*seigniorage*): the printing of new money (notes and coins) is cheap and allows the government to help finance its operations. Today, the monetary policy conducted by central banks is more about maintaining price stability and limiting fluctuations in macroeconomic activity.

In most developed countries up until 2021, the need to control inflation from becoming too high had almost disappeared from public debate. Inflation rates had been maintained at well under 5 percent for three decades. Indeed, concerns often focused on inflation being too low, with episodes nearing deflation during the Great Recession and its aftermath. Then quite suddenly, following the Coronavirus pandemic and Russia's 2022 invasion of Ukraine, inflation rates were well above 5 percent. Today, inflation control is again an important topic in developed economies.

Meanwhile, many less developed and emerging-market economies have struggled with inflation rates remaining at high levels, disrupting daily life in major ways. When inflation runs completely out of control, we speak of *hyperinflation*, which may seem merely like an intellectual curiosity to those who were fortunate enough not to have experienced one, but which quite clearly is disastrous for an economy. Thus, first-order questions for this chapter include: what determines the price level and causes inflation, how is human welfare affected by inflation, and how can stable prices be maintained?

These questions fundamentally concern the "dollar bill" or *fiat money* more generally.

Why does an intrinsically useless piece of paper have value and how do markets and government policy jointly determine its value? We will therefore introduce theories *of* money to explain the role money has in the economy. These theories satisfy Neil Wallace's *dictum* (see Wallace, 1998) that in a theory of money, the use of money should not be an assumption, but an outcome. We will also discuss less ambitious theories—they do not attempt to explain money's role—but are at least theories *with* money that are frequently used. Note that the classification of theories *of* or *with* money is distinct from the usual undergraduate textbook classification of money's three roles as a store of value, a medium of exchange, and a unit of account. With these frameworks, we will discuss the determination of inflation and discuss various options for monetary policy.

A particularly salient phenomenon in monetary economics is *indeterminacy*: the idea that there are many (competitive) equilibria in a given economic environment and that these equilibria may be associated with different real allocations. In a basic sense, this should perhaps not be surprising: money's value today ought to depend on its value in the future, which in turn depends on its value after that, and so on. Can hyperinflations, for example, simply be unfortunate equilibria in economies where there is also an equilibrium with a stationary price level? We will see that in many economic environments, the answer is yes. Thus, it seems that at least in theory, the view held by many, including Milton Friedman, that (hyper)inflation can only result as a consequence of the central bank allowing the money supply to increase (a lot), is not correct.[1] Whether hyperinflations are associated only with excessive money growth empirically is, however, not the focus of this chapter: the main purpose here is merely to cover basic monetary theory.

The need for theory with (or of) money in monetary economics is not a foregone conclusion, however: in the framework that is most frequently employed in policy-making settings today, the New Keynesian model developed in Chapter 18, money is not even present. That setting hence abstracts from monetary aggregates and instead focuses entirely on the role of nominal stickiness (of prices and wages) and how it can make monetary policy have significant real effects. In contrast, in the present chapter, prices will always be assumed to be flexible, so that the discussion can be focused sharply on more basic issues. However, we will demonstrate how the New Keynesian model without money can be motivated, namely, as the "cashless limit" of a model economy with money.

The chapter begins in Section 17.2 with the first fundamental model *of* money: the overlapping-generations model from Chapter 5, for which Samuelson (1958) made the case that intrinsically useless paper money—*fiat* money—could have value under some circumstances. The idea here is that in the overlapping-generations setting, when the young's endowments and preferences are such that they want to save, in the absence of capital or other assets, there is no one they can lend to. However, the presence of fiat money can allow this saving. When fiat money has value, it is therefore an example of an asset bubble: money is an asset, without a dividend, and when people accept money in exchange for goods and services, it must be that people believe in its value only because others do. In this model, money functions as a *store of value*, and it has value so long as there is no other asset that

[1] Friedman (1963)'s famous speech in India includes the assertion "inflation is always and everywhere a monetary phenomenon." Friedman's message, which was based on empirical observation, was that money printing, i.e., central bank policy, is a necessary and sufficient condition for inflation to occur.

dominates money in return, e.g., bears interest. Moreover, the section demonstrates that there is equilibrium indeterminacy: apart from an equilibrium with valued money and a stable price level, there are also hyperinflationary equilibria as well as an equilibrium where money never has value.

Next, Section 17.3 looks at models with infinitely lived agents. There, we first show that money cannot have value, even when other assets are missing. The intuition behind this result is as follows. In a finite-horizon economy, money quite trivially cannot have value in the very last period, since at that time no seller of goods or services will accept money as payment. By induction, it is not valued in earlier periods either. Subsection 17.3.1 shows that this intuition survives also with an infinite horizon but, most importantly, the section sets up a general consumer budget constraint with money as well as government bonds. Next, we move to models *with* money, where a reduced-form liquidity value of money is assumed directly. Subsection 17.3.2 thus begins with the cash-in-advance model where money is assumed to be needed to purchase goods. The motivation is money's second function: the medium-of-exchange role. Similarly, in money-in-transactions-costs and money-in-the-utility-function models, real money balances are assumed to save on transactions costs and to give direct utility, respectively. These models feature rate-of-return dominance: other assets, that bear interest, do not yield these kinds of liquidity services (by assumption). The simple cash-in-advance model we present can also be seen as a motivation for the *quantity theory*: real balances are proportional to real output. Not only undergraduate textbooks but also some advanced research papers use a demand for money formulation that is simply an assumed quantity equation.[2]

Using the reduced-form models, in Subsection 17.3.3 we then briefly discuss optimal monetary policy (absent any stabilization concerns), including the well-known *Friedman rule*. We also show that models incorporating reduced-form liquidity can likewise exhibit equilibrium indeterminacy. We then move to several other important conceptual topics. One is the cashless limit mentioned above: the idea that it is possible to use nominal quantities without even having a money stock present in the model. One of the main purposes of this subsection is to prepare the ground for Chapter 18 on New Keynesian models, where we assume that money is absent but that prices are sticky in nominal units. The cashless limit is non-trivial and involves the notion of *monetary policy rules*, such as interest-rate rules. In particular, when the interest rate is set as an increasing function of the price level, a range of equilibria are eliminated. This insight underlies the use of *Taylor rules* in New Keynesian models and in practical policy-making, where interest rates respond to the price level or the inflation rate. Finally, yet another policy rule that has received significant attention involves the interaction between fiscal and monetary policy: the *fiscal theory of the price level*. We explain its origin and the intuition behind why it can be seen as a mechanism for eliminating equilibria and, yet, is controversial.

We then look at multiple currencies, that is, exchange rates, in Section 17.4. The idea is not to venture into international economics; rather, we discuss the implications of our basic theories of money for the relative values of different fiat currencies that might even circulate within a given economy. This section therefore also involves a short discussion of crypto-currency.

[2]See, e.g., Mankiw and Reis (2002).

In the remaining two sections of this chapter we briefly discuss two fundamental theories of money for which valued money is not an assumption but an outcome of the particular environment. The first approach builds on money as a store of value in an environment with infinitely-lived agents and a limited set of other assets, Section 17.5. In Section 17.6, we look at theories *of* money as a medium of exchange. Here, the idea is that search frictions among traders of goods/services with an *absence of double coincidence of wants* can be seen as a deep friction motivating a value for fiat money.

17.2 Money in overlapping-generation models

Historically, various kinds of currencies have circulated, often in the form of real objects of intrinsic value (such as precious metals), with a variety of price stability and longevity outcomes. However, today money—as defined by notes and coins—is fiat, i.e., it has no intrinsic value, and it is not "backed" (say, by gold). Rather, people voluntarily choose to accept money as a means of payment for goods and services because they expect money to have real value later when they want to use it. Money is thus an asset, but not one that promises anything real of direct value. Asset pricing, as discussed in Chapter 15, should thus be useful, but we need to specify more clearly what the potential future benefits of money might be for someone considering accepting it today.

In this section, we look at money as a store of value. For that, we begin with the overlapping-generations model, which already Samuelson (1958) realized could be used to explain why money—under some conditions—will be valued in equilibrium, despite being fiat. Thus, valued money is an outcome, not an assumption. In fact, there is another plausible outcome where money has no value because people do not believe it will ever have value. Let us now revisit the overlapping-generations model considered in Chapter 5. We proceed using simple examples, which can be easily elaborated on and extended.

17.2.1 An endowment economy

So, first, consider an endowment economy without production or storage. Each period a new cohort enters that lives for two periods and there is a representative agent per cohort. The endowment vector of any agent entering the economy at time 0 or later is (ω_y, ω_o), and the agent's preferences are assumed to be logarithmic: $\log c_y + \log c_o$. The initial old representative agent at time 0 has endowment ω_o and utility that is strictly increasing in c_o. Thus, the environment is stationary.

There is fiat money in the environment: an amount M of perfectly divisible and intrinsically useless objects. We assume that the initial old agents own these. The core question now is whether fiat money can have value. The idea here is that money can be used as a *store of value*. As such, it may be valuable since the endowment economy does not allow saving (and the young have no one to lend to who will be able to pay back).

Let $p_{m,t}$ denote the time t value of a unit of money in terms of consumption goods at time t. Thus, money has value if $p_{m,t} > 0$. But if $p_{m,t} = 0$ for all t, we have a *non-monetary* equilibrium, where no one values money. Then, the maximization problem of the generation

t agent is

$$\max_{c_y, c_o, M'} \log c_y + \log c_o \tag{17.1}$$

subject to

$$c_y + p_{m,t}M' = \omega_y, \quad c_o = \omega_o + p_{m,t+1}M', \quad \text{and} \quad M' \geq 0.$$

The last inequality is natural: money can only be held in positive amounts. This is unlike other assets we discussed so far; for example, we assumed that one can borrow by issuing bonds, that is, holding negative amounts of bonds. The agent of generation -1 makes a trivial decision and simply sets $c_{o,0} = \omega_o + p_{m,0}M$. Recall that the initial old agents begin with the entire stock of money M.

We will continue to assume, for now, that money has value, that is, $p_{m,t} > 0$. We can then combine the constraints from (17.1) to describe the available budget set for the consumer, without explicitly involving money. This delivers

$$c_y + \frac{c_o}{p_{m,t+1}/p_{m,t}} = \omega_y + \frac{\omega_o}{p_{m,t+1}/p_{m,t}} \quad \text{and} \quad \omega_y - c_y \geq 0. \tag{17.2}$$

The implied budget set is presented in Figure 17.1; the inequality constraint, which reflects non-negative money holdings, means that consumption when young cannot exceed endowments when young. As can be seen, the gross real return on money, $p_{m,t+1}/p_{m,t}$, is the slope of the budget constraint.

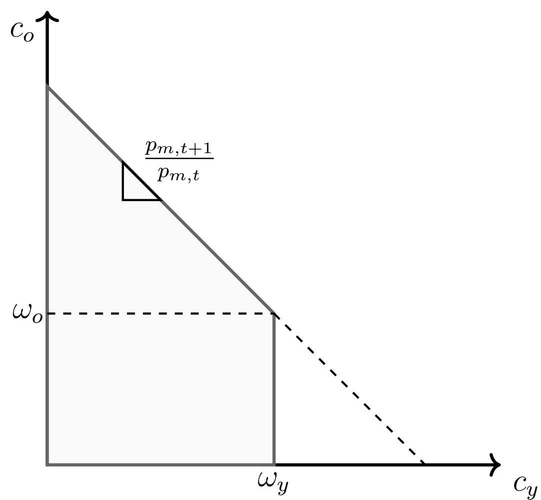

Figure 17.1: Budget set in the economy with fiat money

Solving the maximization problem is straightforward. Since the consumer's indifference curves are strictly convex and strictly decreasing we realize from the figure that either the solution is a point on the downward-sloping budget line that is tangent to the indifference curve—and, hence, the inequality constraint is slack—or it is right at the kink of the budget set, with $c_y = \omega_y$.[3] To find out which case applies, first ignore $\omega_y - c_y \geq 0$ and solve the

[3] Alternatively, use Kuhn-Tucker optimization; here, the Kuhn-Tucker multiplier will be zero in the former case and positive in the latter.

first-order conditions combined with the budget. The solution becomes

$$c_y = \frac{1}{2}\left(\omega_y + \omega_o \frac{p_{m,t}}{p_{m,t+1}}\right)$$

and

$$c_o = \frac{1}{2}\left(\omega_y + \omega_o \frac{p_{m,t}}{p_{m,t+1}}\right)\frac{p_{m,t+1}}{p_{m,t}}.$$

Now check to make sure that $c_y \leq \omega_y$. This amounts to

$$\omega_y \geq \omega_o \frac{p_{m,t}}{p_{m,t+1}} \quad \Longleftrightarrow \quad \frac{p_{m,t+1}}{p_{m,t}} \geq \frac{\omega_o}{\omega_y}.$$

That is, if the gross real return on money is at ω_o/ω_y or above, the solution is valid. The smaller is the ratio ω_o/ω_y, the larger is the consumer's desire to smooth consumption over time and save, but of course, the consumer is also dissuaded from saving if the return on money is low. If $p_{m,t+1}/p_{m,t} < \omega_o/\omega_y$, the solution is $c_y = \omega_y$ and $c_o = \omega_o$ (and $M' = 0$): the consumer would want to buy a negative amount of money (borrow), but cannot. In sum, individual money demand by the young at t as a function of the money prices at t and $t+1$ is

$$p_{m,t}M'_t = \max\left\{\frac{1}{2}\left(\omega_y - \frac{\omega_o}{p_{m,t+1}/p_{m,t}}\right), 0\right\}. \tag{17.3}$$

Given the demand function, we can solve for equilibrium, which amounts to the young buying the entire money stock from the old:

$$M'_t = M \;\forall\; t. \tag{17.4}$$

There are two cases to consider. In one, money has no value at any point in time. This amounts to $p_{m,t} = 0$ for all t.[4] Recall that, in this case, (17.2) is not valid. Instead, the consumer is simply unable to save because no matter how much money they acquire, it will be worthless when they are old. Hence, there is an equilibrium where money does not have value.

If $p_{m,t}$ is instead positive so that money has real value, then we can combine the expression for money demand (17.3) with the money market clearing condition (17.4) and solve for next period's price of money as a function of the current period's price,

$$p_{m,t+1} = \frac{\omega_o p_{m,t}}{\omega_y - 2Mp_{m,t}}. \tag{17.5}$$

This is a non-linear first-order difference equation where the initial value $p_{m,0}$ is not given: it is endogenous (this is the whole point!). Thus, if we can find a solution to this difference equation where $p_{m,t}$ is positive at all points in time, we have a monetary equilibrium. For this, consider the following two cases: (i) $\omega_y > \omega_o$, and (ii) $\omega_y \leq \omega_o$. First, we can see that there is a constant solution to this difference equation: $p_{m,t} = \bar{p}_m = (\omega_y - \omega_o)/(2M)$. This value is positive for case (i), but it is not for case (ii).

Is the constant solution of case (i) the only possible solution or could there be non-stationary equilibria, where the price level changes over time? We plot the dynamics of equation (17.5) for cases (i) and (ii) in the two panels of Figure 17.2.

[4]In this case, $M'_t = M$ can still be assumed to hold: if money is free (and of no use), we can assume it is passed on in full from generation to generation.

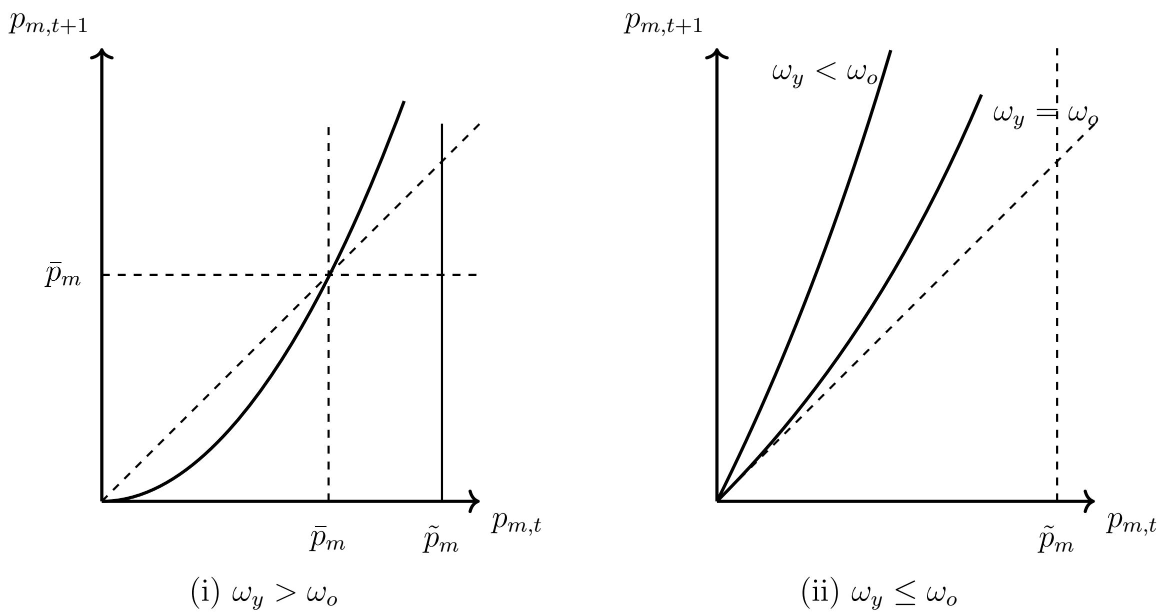

Figure 17.2: Price Dynamics

In Figure 17.2 (i) we plot the difference equation for case (i). The difference equation is defined only for $p_{m,t} < \tilde{p}_m = \omega_y/2M$, since for any current price equal or greater than $\tilde{p}_m$ next period's price is infinite or negative. On the interval $[0, \tilde{p}_m)$, the difference equation is monotonically increasing and it has two steady states at $p_{m,t} = 0$ and $p_{m,t} = \bar{p}_m$. The slope of the difference equation at the two steady states is less than one if money has no value and greater than one if money has value. Now consider an initial price for money $p_{m,0} > \bar{p}_m$. Following the kind of Solow-picture dynamics from Chapter 3, no such value can be part of an equilibrium: sooner or later, $p_{m,t+1}$ will exceed $\tilde{p}_m$, which cannot be an equilibrium. On the other hand, if $p_{m,0} < \bar{p}_m$, we see that the price dynamics will involve monotonically decreasing prices that converge to zero. Thus, money will be steadily losing value and have no value in the limit.

Case (ii), on the other hand, is one in which no monetary equilibrium exists. We plot the difference equation for case (ii) in Figure 17.2 (ii), and it is again increasing on the interval $[0, \tilde{p}_m)$. But now there is only one steady state at the origin when money has no value. Furthermore, the slope of the difference equation at the origin is equal (greater) one when $\omega_o = \omega_y$ ($\omega_o > \omega_y$) and increases in the current price. Thus, for any positive $p_{m,0}$, $p_{m,t+1}$ will eventually exceed $\tilde{p}_m$, although how long this takes depends on how high a $p_{m,0}$ is selected.

To summarize. If $\omega_y > \omega_o$, there is a continuum of equilibria indexed by an initial price level $p_{m,0} \in [0, \bar{p}_m]$. Of these different initial values of money, only one, $p_{m,0} = \bar{p}_m$, gives money a positive value in the limit, and for all the others, the value of money goes to zero. And if $\omega_y \leq \omega_o$, the only equilibrium is one where money has no value ever, $p_{m,t} = 0$.

Finally, note that the number of nominal units in the economy does not have a real importance: It is possible to rewrite the equilibrium difference equation (17.5) as $m_{t+1} = \omega_o m_t / (\omega_y - 2m_t)$, where $m_t \equiv p_{m,t} M$. Thus, the equilibrium determines the real value of the total money stock, independently of how many fiat money units are available.

17.2.2 Welfare comparisons across equilibria

In this model, money derives its value from its role as a store of value, which would otherwise be missing. By fulfilling this function, money can help agents smooth consumption and provides a welfare benefit.

Recall the welfare characterization of overlapping-generations equilibria from Chapter 5: the Pareto efficiency of equilibria depends on the asymptotic marginal rate of substitution between consumption when young and when old or, equivalently, the gross marginal return on saving between consecutive periods. Namely, if this return is above or equal to one, the equilibrium is efficient, and if it is below one, it is inefficient.[5] Hence, in the $\omega_y \leq \omega_o$ case, where money cannot have value, there is Pareto efficiency. When money can have value ($\omega_y > \omega_o$), all equilibria (non-stationary monetary and non-monetary) are Pareto inefficient, except for the stationary monetary equilibrium: the long-run return on saving is $\omega_o/\omega_y < 1$ in the former cases and 1 in the stationary monetary equilibrium.

Is it the case that monetary equilibria, when they exist, are helpful from a welfare perspective? In particular, do they provide a Pareto improvement on the autarky allocation? Clearly, the stationary monetary equilibrium does: the initial old are happier in it, since they can sell their money for a positive amount, and all the other cohorts obtain utility equal to $2\log((\omega_y + \omega_o)/2)$, which exceeds $\log \omega_y + \log \omega_o$. Notice that the fully smoothed allocation is better than autarky for all cohorts born at $t \geq 0$, also in the case where $\omega_y < \omega_o$, but money cannot help attain this allocation. Moreover, if it could, it would make the initial old worse off.

In fact, all the non-stationary monetary equilibria do improve on autarky as well. Moreover, the monetary equilibria are ranked in the following sense: the larger is $p_{m,0} \leq \bar{p}_m$, the higher is utility for all generations. We will leave these results as an exercise for the reader; the key insight is that a higher return on saving is better for the consumer in this case.

17.2.3 Extensions: a neoclassical growth economy and policy

The basic results of the previous section extend in various ways. The essential ingredient for money to have value in this context is the lack of a (good) store of value. Alternatively, money does not have value here, if people don't need a store of value—such as when $\omega_y < \omega_o$—or a "good enough" alternative store of value is available. So, consider the overlapping-generations version of the neoclassical growth economy of Chapter 5, where people can save using physical capital. Capital would then compete with money and potentially rule out monetary equilibria.

Now, the characterization of equilibria is somewhat more cumbersome than in the endowment case. However, consider any such equilibrium, and for the preferences assume $\log c_y + \log c_o$, just as above. Recall that an equilibrium is still Pareto inefficient if, as time goes to infinity, the limit for the gross interest rate is below one. This, moreover, is an outcome that is possible in the neoclassical model. It's another good exercise to try to derive a

[5] The non-monetary equilibria described here do not have a return on saving since there is no active savings vehicle. However, we can introduce borrowing and lending, which have to equal zero in equilibrium. The gross interest rate therefore needs to adjust to ω_o/ω_y to make the young choose exactly zero borrowing.

condition under which you can see this to be true.[6] In such a case, fiat money can have value and real effects on economic activity. Capital will not be displaced entirely if production is Cobb-Douglas, so money and capital will both be used and give the same gross return of 1 asymptotically.

It is straightforward to introduce money printing into the model, e.g., as lump-sum transfers. If the government increases the money supply at a gross rate of $1+\mu$ every period, then the same type of equilibrium characterization as above obtains, with the difference that the stationary monetary equilibrium is now replaced by one where the real return on money is below one—it is $1/(1+\mu)$—and where the total value of the money stock is lower. Money printing can also be used as a source of income for the government (seigniorage) to pay for its expenditures or debts. One can thus also introduce government bonds easily. If they are in positive supply, bonds would compete with money as a store of value. Open market operations can be studied, whereby the government would change its outstanding stock of bonds over time, which in general will also affect allocations and the value of money. However, the effects on allocations only materialize if the present-value budget of a cohort is changed; otherwise, a Ricardian-equivalence theorem applies, as studied in Chapter 15. We will return to some of these issues in the next section.

Thus we can construct versions of the overlapping-generations model for which valued money co-exists with alternative means of storage if it pays the same rate of return as these alternative assets. However, the overlapping generations model still has trouble explaining why people in actual economies hold money despite there being other assets that dominate it in return.

17.3 Money in dynastic models

Dynastic models are different from overlapping-generations models in that their competitive equilibria, absent other frictions, are Pareto optimal. Thus, at least in this sense, money is not needed (as a store of value or otherwise). We will thus begin to demonstrate that indeed, money cannot have value in the basic dynastic settings. Then we introduce, one by one, additional assumptions that will give money value.

17.3.1 Fiat money has no value

We now show that in a standard infinite-horizon environment with dynastic households fiat money has no value if it only serves as a store of value. For this and the following sections, we will consider variations of a representative agent production economy with labor or one with endowments only (both covered in Chapter 5). Unlike in the section on overlapping generations, we will now explicitly consider a more general structure with government bonds as well as changes in the stock of money.

The household's preferences over consumption, c, and leisure, l, and the production of

[6] To find at least one example, look at a case where there is a closed-form solution, such as when $\omega_o = 0$ and δ, the rate of depreciation of physical capital, is 1.

consumption through labor are described by

$$\sum_{t=0}^{\infty} \beta^t u\left(c_t, l_t\right) \tag{17.6}$$

$$c_t = z\left(1 - l_t\right). \tag{17.7}$$

We have assumed a fixed time endowment of one.[7] The planning solution would hence maximize the stated utility subject to this constraint; the solution is time-independent and characterized by $zu_c(c, l) = u_l(c, l)$.

The household's budget constraint, in real terms, is

$$c_t + q_t a_{t+1} + p_{m,t} M_{t+1} + p_{m,t} B_{t+1} = w_t\left(1 - l_t\right) + a_t + p_{m,t} M_t + p_{m,t}\left(1 + i_{t-1}\right) B_t - \tau_t, \tag{17.8}$$

where a is a real asset with discount rate $q_t = 1/(1+r_t)$, and r_t is the net real return between t and $t+1$. Real wages are w, and the government imposes a lump sum tax, τ. M and B are nominal holdings of money and one-period bonds, respectively, denominated in units of money, and p_m is the price of money. Hence, an equilibrium where money has no value would have $p_{m,t} = 0$ for all t. The nominal net interest rate on bonds between t and $t+1$ is i_t.

The household is allowed to borrow or lend in real assets and nominal bonds, but cannot borrow by issuing money,

$$M_t \geq 0. \tag{17.9}$$

The government's budget constraint (where we abstract away from real expenditures) is given by

$$p_{m,t}(M_{t+1} - M_t) + p_{m,t} B_{t+1} + \tau_t = p_{m,t}\left(1 + i_{t-1}\right) B_t. \tag{17.10}$$

In equilibrium, the representative household holds outstanding fiat money and nominal government debt and real assets are zero, $a_t = 0$. Thus, the resource constraint is satisfied: $c_t = z(1 - l_t)$.

Let $m_t \equiv p_{m,t} M_t$ and $b_t \equiv p_{m,t} B_t$ denote the real value of money and bonds and define $\hat{a}_t = a_t + m_t + b_t(1 + i_{t-1})$ as total real wealth at the beginning of the period. By using this definition, along with some algebra, we can rewrite the household's budget as

$$c_t + q_t \hat{a}_{t+1} + \left[\frac{p_{m,t}}{p_{m,t+1}} - q_t\right] m_{t+1} + \left[\frac{p_{m,t}}{p_{m,t+1}} - q_t\left(1 + i_t\right)\right] b_{t+1} = w_t\left(1 - l_t\right) - \tau_t + \hat{a}_t. \tag{17.11}$$

In this budget constraint, $\hat{a}$ (and no other variable) is used to save from t to $t+1$. The terms involving m_{t+1} and b_{t+1} are "static": they are losses (gains) at t to the extent the expressions in square brackets are positive (negative). Thus, first, the household would not want to hold money if its real return, $p_{m,t+1}/p_{m,t}$, is below the real interest rate $1/q_t$. Second, the household would be able to attain unbounded consumption for given wealth and prices if the real return on bonds, $(1 + i_t)p_{m,t+1}/p_{m,t}$ were not equal to the real interest rate: if the return is higher (lower), the consumer could obtain unbounded resources by raising

[7] Note that in the other chapters of this textbook, we use ℓ for labor supply. In the context of this chapter, we have $l + \ell = 1$.

(lowering) b_{t+1} without bound. Thus, for an equilibrium with positive money and finite bond holdings to exist, the following needs to hold:

$$q_t = \frac{p_{m,t}}{p_{m,t+1}} = \frac{p_{m,t}}{p_{m,t+1}} \frac{1}{1+i_t}. \tag{17.12}$$

These no-arbitrage conditions mean that we can write the budget constraint in more compact form as

$$c_t + q_t \hat{a}_{t+1} = w_t (1 - l_t) - \tau_t + \hat{a}_t. \tag{17.13}$$

We see immediately from equation (17.12) that money cannot be held in positive amounts—it cannot have real value, i.e., p_m will have to equal 0—if the nominal interest rate on bonds is positive. If $i_t > 0$, money would then be dominated in return by bonds and be a pure loss to hold in positive amounts. Second, if i_t is zero at all times (or nominal bonds are not available in the economy), then money (or bonds, which are now equivalent to money) cannot have value either.[8] To see this, note that the consumer's optimization problem collapses to a problem with one asset with real return $1/q_t$, equation (17.13). From Chapter 4 we know that for this problem the transversality condition is a necessary condition for optimality. Applying the condition to the real value of money holdings we obtain

$$0 \geq \lim_{T\to\infty} q_0 q_1 \ldots q_T m_T = \lim_{T\to\infty} \frac{p_{m,0}}{p_{m,1}} \frac{p_{m,1}}{p_{m,2}} \ldots \frac{p_{m,T-1}}{p_{m,T}} p_{m,T} M_T = p_{m,0} \lim_{T\to\infty} M_T. \tag{17.14}$$

Thus, if fiat money is not vanishing (i.e., being withdrawn) in this economy, money cannot have value earlier on. In other words, for money to have value in this economy it must disappear in the limit. We will return to this point.

So far we have been interested in whether fiat money can have value, that is, whether the price of money in terms of goods can be positive. In the following we will mainly deal with environments for which money does have value, that is, $p_m > 0$. For these environments we are often interested in the price of goods in terms of money, that is, the price level, $P = 1/p_m$, and the rate at which the price level is changing, that is, the gross inflation rate, $1 + \pi = P'/P = p_m/p'_m$. For nominal bonds, this means that we will frequently write the expression for their return as the *Fisher equation*:

$$1 + i_t = (1 + r_t)(1 + \pi_t), \tag{17.15}$$

which states that the gross nominal interest rate equals the gross real interest rate times the gross inflation rate. In our perfect foresight environment, the Fisher equation can be seen as an arbitrage condition involving real and nominal bonds.

17.3.2 Fiat money with reduced-form liquidity services has value

There are different ways to give value to money by assuming, in a reduced-form way, that it matters to consumers. We now briefly look at them, one by one. Because government bonds, from the perspective of the consumer, are identical to borrowing and lending—both b and a can be held in positive as well as negative amounts and hence will yield the same return—we will only keep a. We will sometimes refer to the nominal interest i_t, which as before means the money return at $t+1$ on a nominal bond bought at t.

[8] Recall that nominal interest rates at zero were observed for an extended period in the aftermath of the Great Recession, so this case is not just a theoretical one.

The cash-in-advance model

We first impose the constraint that money has to be used in transactions; in particular, we assume that goods can only be purchased using money. This constraint, commonly known as a cash-in-advance (CIA) constraint, from Clower (1967), gives rise to an equilibrium where money has value and is dominated in return by other assets.

Let us first incorporate the CIA constraint into the household's budget constraint:

$$c_t \leq p_{m,t} M_t \tag{17.16}$$

$$q_t a_{t+1} + p_{m,t} M_{t+1} = a_t + p_{m,t} M_t - c_t + w_t \left(1 - l_t\right) - \tau_t. \tag{17.17}$$

The first expression is the CIA constraint and states that money is needed to buy goods. The second expression states that money that is not spent today can be saved, holding money or the asset a. The two expressions combined also state that current earnings are not available contemporaneously for consumption.

Consider now the modified consumption-savings problem of the representative agent

$$\max_{\{c_t, l_t, a_{t+1}, m_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u\left(c_t, l_t\right)$$

subject to

$$c_t \leq m_t \tag{17.18}$$

and

$$c_t + q_t a_{t+1} + \frac{p_{m,t}}{p_{m,t+1}} m_{t+1} = a_t + m_t + w_t \left(1 - l_t\right) - \tau_t, \tag{17.19}$$

where we have again replaced nominal money balances with real balances. The first-order conditions for the household's problem are

$$u_1\left(c_t, l_t\right) = \lambda_t + \mu_t, \tag{17.20}$$

$$u_2\left(c_t, l_t\right) = \lambda_t w_t, \tag{17.21}$$

$$\lambda_t q_t = \beta \lambda_{t+1}, \tag{17.22}$$

and

$$\lambda_t \frac{p_{m,t}}{p_{m,t+1}} = \beta\left(\lambda_{t+1} + \mu_{t+1}\right), \tag{17.23}$$

where $\beta^t \mu_t$ and $\beta^t \lambda_t$ are the Lagrange multipliers on the CIA constraint (17.18) and budget constraint (17.19), respectively, and $u_i(\cdot, \cdot)$ is the partial derivative with respect to the ith argument.

Now consider a stationary equilibrium for a constant money stock M and labor productivity, z. In the equilibrium the household holds all of the money, net assets are zero, $a = 0$, consumption and leisure satisfy $c = z(1 - l)$, and the real wage equals the marginal product of labor, $w = z$. From the FOCs for consumption and leisure, it follows that the Lagrange multipliers are constant, and from the FOC for assets, it follows that the discount rate on assets is equal to the discount factor, $q = \beta$.

If the CIA constraint is binding with a constant c and M, then the price of money is constant at $p_m = c/M$. As there is no inflation, the nominal interest rate satisfies $q(1+i) = 1$. From the FOC for real balances, it then follows that the Lagrange multiplier on the CIA is positive, $\mu > 0$. Combining the FOC for consumption and leisure, we obtain

$$\frac{u_2(c, 1 - c/z)}{u_1(c, 1 - c/z)} = z\frac{\lambda}{\lambda + \mu} = \beta z, \tag{17.24}$$

where we have also used the Euler equation for real balances, (17.23). Equation (17.24) determines the steady-state consumption level from which all remaining equilibrium variables can be obtained.

Notice that the CIA must be binding in the stationary equilibrium. To see why, suppose it is not binding, that is, $c < p_{m,t}M$ and $\mu = 0$. From the first-order condition for real balances, it then follows that the return on money is equal to the interest rate. But this implies that the price of money is increasing over time. So the CIA constraint remains satisfied, but the TVC is now violated, as shown in the previous section.

Also, notice that compared to the same economy without the CIA constraint, the marginal rate of substitution between leisure and consumption is less than the marginal rate of transformation: $\beta z < z$. But this means that in the CIA economy leisure is higher and consumption is lower. The real wage effectively received by the household is lower in the CIA economy because today's labor income can only be used tomorrow, and the return on saving the labor income is less than the real interest rate. Is there a way to fix the problem?

Suppose the government imposes a nominal lump sum tax on the representative household, T_t, to be paid using money. Also, assume that money is withdrawn at a constant rate, $M_{t+1}/M_t = \gamma < 1$. Then the implied real lump sum tax is $\tau_t = p_{m,t}(M_t - M_{t+1}) = (1 - \gamma)p_{m,t}M_t$. We can still find a stationary equilibrium with a binding CIA constraint where the price of money increases as the money stock shrinks and the value of the total money stock is constant, $M_t p_{m,t} = c$. If the nominal stock changes at a gross rate γ, then the value of each unit of money changes at $1/\gamma$, and the discount rate for real balances is γ; for $\gamma > 1$, this means a constant rate of inflation and for $\gamma < 1$ a constant rate of deflation.

We can, in particular, choose $\gamma = \beta$ such that the return on assets and money is equalized. For this policy the TVC is satisfied since the nominal money stock is vanishing in the limit. This result—that withdrawing money from the economy at the rate of discount makes the equilibrium optimal, and constitutes optimal monetary policy—is known as the *Friedman rule*. Milton Friedman cast this policy in terms of "paying interest on money", which is something a central bank could engineer. With the money stock shrinking at rate β, people are not constrained in the use of money, and similarly if money paid interest and were therefore identical to bonds, they would not be constrained either.

The CIA model of money does not satisfy Wallace's dictum: though we have not demonstrated it, money will always be valued here because it has been hard-wired to be equal to consumption (in real terms, and hence its value could not be zero, or else consumption would be zero). Money is a store of value in the CIA model, but one that is worse than bonds and other assets: its value derives from it being required to buy goods. Money is thus used as a medium of exchange, though the exchange is not explicitly modeled. One way to describe the CIA constraint is that it imposes a *quantity equation* of sorts: the equation $VM = Py = Pc$ is met by assumption with a velocity of 1, $V = 1$.

The quantity theory in the data

The defining characteristics of the Quantity Theory of Money (QTM) are the presence of a stable demand for money and that money growth and nominal GDP growth move one for one. Suppose that money is neutral over the long run, that is, real activity is independent of money growth over the long run. Then price inflation will move one-for-one with money growth in excess of real GDP growth over the long run if the demand for money is stable and proportional to the nominal transaction volume.

For an empirical evaluation of QTM one thus needs measures of the money stock, the transaction volume, quantities and prices, and the opportunity cost of holding money. From the point of view of money as a means to execute transactions, one can think of various measures. Standard measures of money published by central banks include M1, consisting of currency and checkable demand deposits, and M2, which adds savings deposits to M1. Regulatory changes and technical advances may affect what should be included in a measure of money, see Lucas and Nicolini (2015). For example, before the 1980s, Regulation Q in the U.S. imposed limits on the ability of banks to pay interest on accounts. Once Regulation Q was abolished, banks started to offer new interest-bearing liquid accounts, e.g., money market deposit accounts with limited transaction features. In the mid-1990s, IT improvements made moving funds between demand deposits and money market accounts easier (SWEEP accounts), making the latter close payment substitutes. Payments from bank accounts have also been made easier using electronic transfers that can be initiated using mobile phones. Bitcoin and other digital currencies may represent additional future means of payment. Thus what should be included in a measure of money changes over time, which affects the stability of money demand as defined by a fixed measure of money. Furthermore, the M1 and M2 measures listed are simple sums of their various components, but these components may well differ in their ability to perform transactions. Divisia indices have been proposed for constructing aggregate money stocks; see Barnett (1980). This approach is analogous to how we aggregate different final goods into an aggregate output measure like GDP. The standard measure of transactions in the literature is GDP, but obviously, many transactions precede the purchases of final goods in the economy: there are usually multiple stages of production. Thus, using GDP as a sufficient statistic for the transaction volume implicitly assumes that the production structure is not changing much over time. Finally, various short-term interest rates have been used for the opportunity cost of holding money. For the U.S. that includes the Federal Funds rate, the rates on short-term commercial paper or short-term U.S. Treasuries. In the following, we study how well the QTM holds up for the U.S. for the period 1901 to 2023 using M2 as a measure of the money stock, GDP as a measure of transactions, and the 6-month commercial paper rate as a measure of the opportunity cost of holding money. Using other measures of money and interest rates produces very similar results.

We first consider the long-run relation between M2 growth and inflation, following Sargent and Surico (2011). For this purpose, we calculate long-run movements of M2 growth and inflation as 15-year symmetric moving averages. In Figure 17.3 we plot the filtered M2 growth rates and inflation for annual data from 1902–2023 for five subsamples: 1902–1928, 1929–1954, 1955–1983, 1984–2005, 2006–2023. The first period precedes the Great Depression, the second period covers the Great Depression and World War II, the third period covers the post-WW-II period including the inflationary 1970s, the fourth period covers what has been called the Great Moderation and the adoption of inflation targeting among advanced economies, and the fifth period covers the Great Recession and the policy of Quantitative Easing (QE). We see that inflation moves roughly one-for-one with money growth, but that there is notable variation in that relation across subsamples. On the one hand, during the period covering high inflation in the late 1960s and 1970s, inflation appears to respond strongly to changes in money growth. On the other hand, during the Great Moderation and inflation targeting, the response of inflation to changes in money growth is much weaker, and during the QE period, inflation appears to decline as money growth increases.

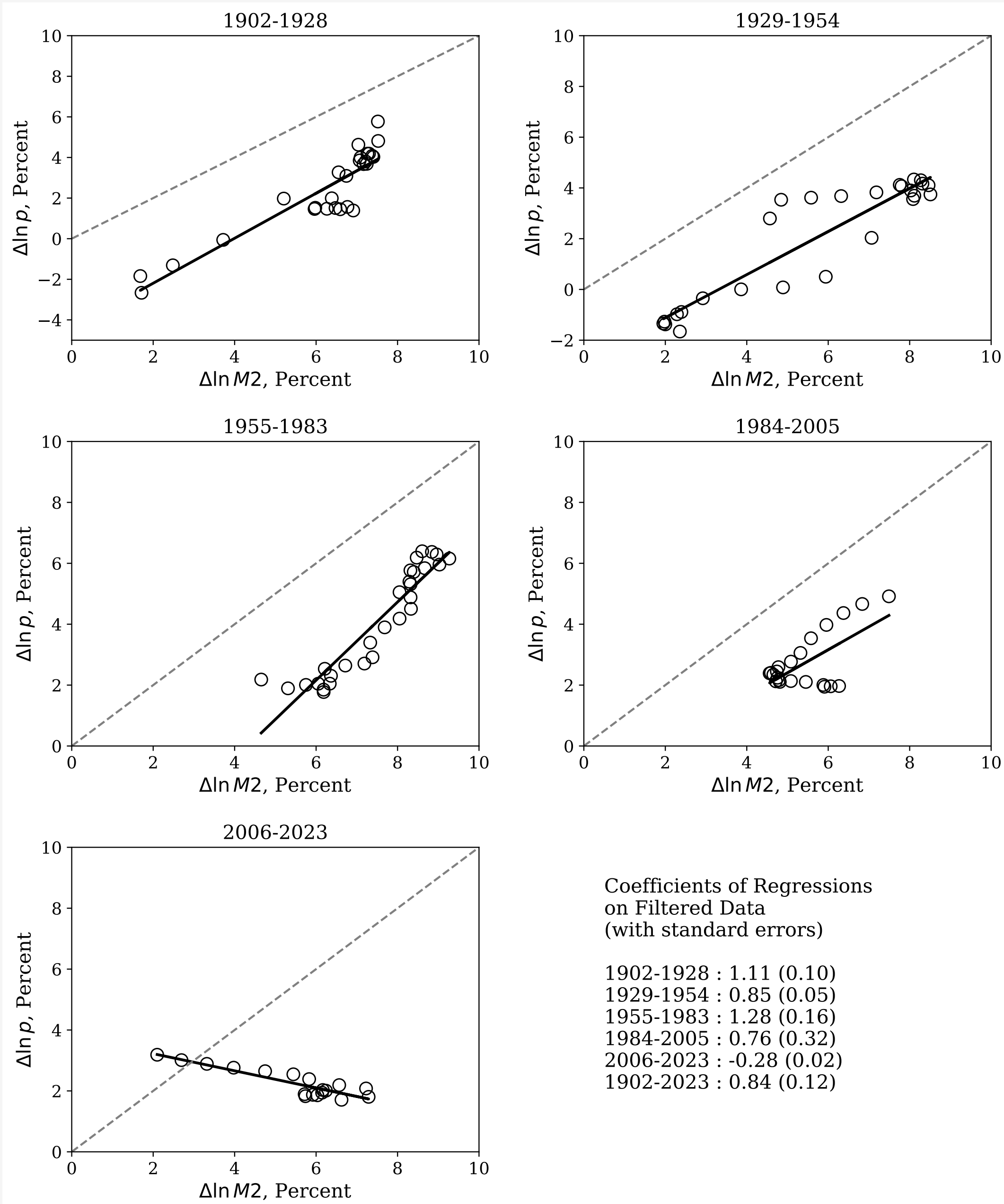

Figure 17.3: Inflation and money growth

Notes: Each panel plots the filtered inflation rates against the filtered M2 growth rates (circles), and contains the 45-degree line (dashed line) and the fitted values from an OLS regression of inflation on money growth (thick line). The estimated OLS coefficients for the sub-samples are in the lower right-hand corner, with heteroskedasticity and auto-correlation corrected standard errors in parentheses.

We now consider the long-run stability of money demand. In Figure 17.4 we plot the M2-GDP ratio and the 6-month commercial paper rate for both the actual data and their 15-year symmetric moving averages. At first inspection, Figure 17.4 seems to provide evidence for the long-run stability of money demand despite the large and persistent deviations of the variables from their long-run trends. Whenever the filtered short rate increases, the filtered M2-GDP ratio declines. This stable relationship for the filtered data disappears, however, when we look at the sub-samples as in Figure 17.3. Now, the OLS regression coefficients of the M2-GDP ratio on the short rate can be either positive or negative, and they are usually not significant. A cautious summary is that the evidence on QTM is mixed.

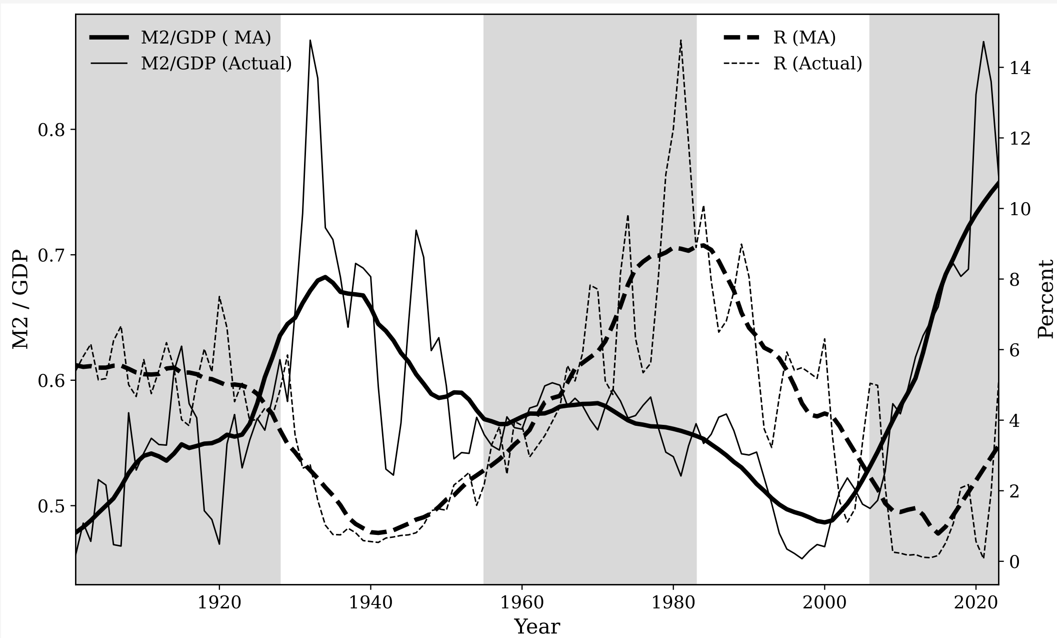

Figure 17.4: Money demand

Notes: The thin lines are the annual data for the ratio of M2 to nominal GDP (solid, left axis) and the 6-month Commercial Paper (CP) rate (dashed, right axis). The thick black lines are the 15-year symmetric moving averages of the M2-GDP ratio (solid) and the CP rate (dashed). Shading and its absence sets apart the five subsamples from Figure 17.3.

Besides a CIA constraint, there are other ways to introduce money into a standard representative agent frictionless economy. Next, we consider two prominent examples. The first example posits that transactions are costly, but less so if one holds money. The second example represents any inherent value to money by incorporating it directly into the utility function.

Money reduces transactions costs

Consider the budget constraint (17.19), but assume that the purchase of consumption goods involves a real cost in terms of the consumption good,

$$c_t + q_t a_{t+1} + \frac{p_{m,t}}{p_{m,t+1}} m_{t+1} = a_t + m_t + w_t (1 - l_t) - \tau_t - \psi \left(\frac{c_t}{M_{t+1}/P_t} \right) c_t, \tag{17.25}$$

where $\psi(x) = \kappa x^{\eta}$ is an increasing function, $\kappa, \eta > 0$. Thus transaction costs are increasing in consumption, decreasing in current real balances, which we recall are $M_{t+1}/P_t = m_{t+1} p_{m,t}/p_{m,t+1}$, and homogeneous of degree one in consumption and real balances jointly. Here, real balances chosen in the current period facilitate transactions, unlike in the CIA model where real balances carried over from the previous period are required for transactions.

Using a subscript t on ψ and its derivative to indicate evaluation at $c_t / (m_{t+1} p_{m,t}/p_{m,t+1})$, we obtain first-order conditions for the household's problem as follows:

$$u_1 (c_t, l_t) = \lambda_t \left[1 + \psi_t' \cdot \frac{c_t}{m_{t+1} p_{m,t}/p_{m,t+1}} + \psi_t \right],$$

$$u_2 (c_t, l_t) = \lambda_t w_t,$$

$$\lambda_t q_t = \beta \lambda_{t+1}, \tag{17.26}$$

and

$$\lambda_t \frac{p_{m,t}}{p_{m,t+1}} \left[1 - \psi_t' \cdot \left(\frac{c_t}{m_{t+1} p_{m,t}/p_{m,t+1}} \right)^2 \right] = \beta \lambda_{t+1}, \tag{17.27}$$

where $\beta^t \lambda_t$ is the Lagrange multiplier on the budget constraint (17.25). Combining the FOCs for the real asset and real balances, (17.26) and (17.27), we obtain

$$1 - \frac{q_t}{p_{m,t}/p_{m,t+1}} = \psi_t' \cdot \left(\frac{c_t}{m_{t+1} p_{m,t}/p_{m,t+1}} \right)^2. \tag{17.28}$$

Using the Fisher equation (17.15) for the nominal interest rate on the left-hand side and using the functional form for ψ on the right-hand side we arrive at an expression,

$$M_{t+1} = P_t c_t \left[\frac{1}{\kappa \eta} \frac{i_t}{1 + i_t} \right]^{-1/(1+\eta)}, \tag{17.29}$$

which relates the demand for money to the value of transactions and the opportunity cost of holding money, the nominal interest rate. This expression is in line with the usual expressions for money demand, unlike in the simple CIA model, for which money demand is interest-inelastic.[9] That is, real money demand is equal to consumption (or output, if output is defined net of transactions costs) times the inverse of velocity, an expression that is decreasing in the nominal interest rate. We have so far described this model only from a single consumer's perspective. To solve for a general equilibrium, we would need to specify assumptions on production, taxes, and market clearing.

[9] More elaborate versions of the CIA model also feature interest-elastic money demand.

Money in the utility function

We have just assumed that real balances are for some reason helpful in executing consumption purchases. We could take an even more reduced form approach and assume that real balances are valuable, period. For this purpose, we simply include real balances in the utility function of the representative agent, rather than modifying the budget constraint. Instead, assume that preferences are

$$\sum_{t=0}^{\infty} \beta^t \left[u\left(c_t, m_{t+1} \frac{p_{m,t}}{p_{m,t+1}} \right) + v\left(l_t\right) \right], \tag{17.30}$$

where the second argument of u is simply the real amount of money purchased at t, M_{t+1}/P_t, but rewritten in terms of the real balances in the beginning of next period and the prices of money in the two consecutive periods. The specific functional form used is convenient but of course not the only possibility. The FOCs for the household's consumption-savings problem with preferences (17.30) and budget constraint (17.19) are

$$u_{1,t} = \lambda_t, \tag{17.31}$$

$$v_t' = \lambda_t w_t, \tag{17.32}$$

$$\lambda_t q_t = \beta \lambda_{t+1}, \tag{17.33}$$

and

$$\lambda_t \frac{p_{m,t}}{p_{m,t+1}} = u_{2,t} \frac{p_{m,t}}{p_{m,t+1}} + \beta \lambda_{t+1}, \tag{17.34}$$

where we have suppressed the arguments of the derivatives of u and v for compactness ($u_{i,t}$ again denotes the partial derivative with respect to ith argument), and subscript t denotes time-dependence of the arguments. Another equation is also relevant: our budget constraint (17.19) does not explicitly feature nominal government bonds paying interest i_t between t and $t+1$, but that is because of the no-arbitrage condition between real and nominal bonds that yields the Fisher equation, $q_t(1+i_t) = p_{m,t}/p_{m,t+1}$. Using this equation together with equations (17.33) and (17.34), we obtain $1 = u_{2,t}/u_{1,t} + 1/(1+i_t)$. This delivers an expression that is very similar to equation (17.29):

$$\frac{u_{2,t}}{u_{1,t}} = \frac{i_t}{1+i_t}. \tag{17.35}$$

If we assume that $u(c,m)$ is a homothetic function, that is, it is a monotone transformation of a homogeneous of degree one function, we again obtain a static money demand equation that relates the ratio of real balances to consumption, m/c, to the opportunity cost of holding real balances, $i/(1+i)$, like (17.29). As in the transactions-costs case, the stationary general equilibrium allocation is then obtained by adding assumptions on production, taxes, and market clearing.

17.3.3 Policy and the value of money in the reduced-form models

We will now illustrate some key features of the reduced-form models of money.

Stationary equilibria: quantity theory and optimality

All of our reduced-form models of fiat money give rise to a well-defined demand for real balances. In the CIA model, this demand for real balances is independent of the nominal interest rate, equation (17.18) with equality. But for the models where real balances reduce transaction costs or provide utility directly, this demand may depend on the nominal interest rate, e.g., equations (17.29) and (17.35). Since the environment in our examples is stationary so far we have studied their stationary equilibria for which quantities and prices are constant. In these equilibria real balances are constant and therefore the price of money is inversely proportional to the money stock. This reflects the quantity theory which is usually taken to state that the price level is proportional to the money stock.

The Friedman rule In the CIA model, we have shown that in general the stationary equilibrium outcome is sub-optimal relative to the economy without frictions, that is, without the CIA constraint. We also saw that the outcome of the frictionless economy can be recovered if the money stock is shrinking at a rate such that the price of money is increasing and the real rate of return on money is equal to the real discount rate. In other words, the price level is falling, and there is deflation at the real interest rate. Furthermore, from the Fisher equation, it then follows that the nominal interest rate is zero. This is the Friedman rule: if money is dominated in return but serves a role in the economy, then optimal policy is to reduce the opportunity cost of holding money to reduce distortions. For the CIA model, such a policy can eliminate the distortion, but for the two other reduced-form models of money, the full optimum can only be reached exactly if additional assumptions about satiation in money holdings are imposed. This means that for some finite value of real balances, transaction costs are zero in the first economy, and preferences have a bliss point in real balances in the second.

Equilibrium indeterminacy

A feature of monetary economies is that they naturally give rise to multiple equilibria. This arises because giving up resources to buy money today is unattractive if money is expected to lose all value tomorrow (i.e. if others are not willing to buy it): the price of money today depends naturally on the price of money in the future. In the overlapping-generations economy of section 17.2, there is always an equilibrium where money never has value because money is used only as a savings vehicle. As we shall see here, reduced-form models often also deliver indeterminacy.

Hyperinflation In the reduced-form models, equilibria where money never has value can occur but only under special assumptions on preferences/transactions costs.[10] There may, however, exist non-stationary monetary equilibria. Consider, for example, the CIA model

[10]In the CIA model, a non-monetary equilibrium would mean zero consumption, which is only possible if utility is bounded below in consumption. In the model where money reduces transaction costs, these costs would need to be bounded at zero real money balances, and in the model where money enters the utility function, this function must be bounded below in real balances.

from section 17.3.2. Combine equations (17.20)-(17.23), assuming that the CIA constraint is binding, and normalize $z = 1$. We obtain

$$u_2(m_t, 1 - m_t)\frac{p_{m,t}}{p_{m,t+1}} = \beta u_1(m_{t+1}, 1 - m_{t+1}). \tag{17.36}$$

Now assume that through lump-sum transfers (negative taxes) the money stock changes at the constant gross rate $\gamma > 1$, $M_{t+1} = \gamma M_t$, and we arrive at a first-order difference equation in real balances, m_t,

$$\gamma u_2(m_t, 1 - m_t)\frac{m_t}{m_{t+1}} = \beta u_1(m_{t+1}, 1 - m_{t+1}), \tag{17.37}$$

with no initial condition for real balances. This makes clear how expectations fundamentally drive equilibrium outcomes. One solution is a steady state, but this may not be the only possibility. To make this point particularly sharp, suppose that leisure and consumption are separable and that the marginal utility of leisure is constant and equal to 1: with a slight abuse of notation, $u(c, l) = u(c) + l$. This yields

$$\gamma m_t = \beta m_{t+1} u'(m_{t+1}). \tag{17.38}$$

If in addition $\lim_{m\to 0} mu'(m) = 0$, then this expression defines two steady states: one with positive real balances, $\gamma/\beta = u'(\bar{m})$, and a limiting steady state where money has no value and real balances are zero. Furthermore, for any initial real balances $0 < m_0 < \bar{m}$ the path of real balances defined by equation (17.38) converges to the steady state with zero real balances.[11] Given M_0 an initial m_0 corresponds to a price $p_{m,0}$, and $p_{m,0} < \bar{p}_{m,0}$ for $m_0 < \bar{m}_0$. Thus, for any initial $p_{m,0} < \bar{p}_{m,0}$ the price of money converges to zero faster than the money stock increases, and the value of real balances vanishes in the limit.[12]

Local indeterminacy In a different environment, the same policy of constant money growth through lump-sum transfers can result in a continuum of nonstationary equilibria that all converge to the unique stationary equilibrium. For a version of the reduced-form model with money in the utility function from section 17.3.2, Obstfeld (1984) shows that the local dynamics of perfect foresight paths at the steady state allow for a continuum of paths that all converge to the steady state if consumption is a normal good and the magnitude of the elasticity of marginal utility of consumption with respect to real balances is sufficiently small. For all of these equilibria, the price level relative to the money stock remains bounded. Below we will study the interaction of monetary and fiscal policy and explore how policy modifications can lead to a unique equilibrium.

[11] Linearizing the dynamic system around $m = \bar{m}$, we see that the system is locally unstable. Linearizing around $m = 0$, however, delivers stability, provided marginal utility is sufficiently large, $u'(0) > \gamma/\beta$.

Note also that along paths where $m_t \to 0$, the real return on money is always less than in the stationary equilibrium with positive real balances, and thus the CIA is binding.

[12] The example in this section relies on specific assumptions on utility. Under other assumptions, there is a unique equilibrium. If $u(c) = \log c$, which violates the condition stated as $\lim_{m\to 0} mu'(m) = 1 > 0$, equation (17.38) allows us to solve uniquely for m_0, m_1, and so on.

Different monetary rules

The above discussion contains an implicit assumption about the conduct of monetary policy: the monetary authority selects a path for the money supply, a sequence that will be given to the economy, and for which one can then examine the set of implied equilibrium allocations. We learned—for the economies we looked at—that for a money supply path featuring growth at a constant rate, there is a unique steady-state equilibrium in which the rate of inflation equals the inverse of the money growth rate. We also learned that other equilibria may exist under some conditions. For the case in which there are government bonds, the nominal interest rate consistent with the money stock sequence would follow from the Fisher equation. However, in most economies of today, monetary policy is not described this way; rather, the more appropriate description of the conduct of monetary policy is that of a "choice of a sequence of nominal interest rates", possibly associated with a rule such as the Taylor rule as described below. We now discuss how to define equilibria where the central bank directly chooses interest rates (and the money supply path becomes endogenous).

Interest rate rules

Sims (1980) popularized the use of vector autoregressions (VARs) as a way of studying the effects of exogenous shocks on macroeconomic time series while imposing minimal assumptions to identify the exogenous shocks (see Chapter 8). Early VAR applications that studied monetary policy as a source of economic fluctuations were influenced by QTM arguments. Sims (1972), in a small-scale VAR of the U.S. economy with real GDP and nominal money finds a quantitatively large contribution of money shocks to output fluctuations. Later work considered larger-scale VARs, and once short-term interest rates were added to the list of variables, interest rate shocks replaced money shocks as a source of output fluctuations. Bernanke and Blinder (1992) then argued that monetary policy actions set a particular short-term interest rate, namely the Federal Funds rate, the overnight interest rate in the market for interbank loans, and thus interest rate shocks reflect the impact of monetary policy. Contemporaneously, Goodfriend (1991), based on a reading of U.S. monetary policy implementation at the Federal Reserve System, argues that

> "Except for the period from 1934 to the end of the 1940s when short-term interest rates were near zero or pegged, the Fed has always employed either a direct or an indirect Federal Funds rate policy instrument." (p.8)

In particular, he observes that the Federal Funds rate is adjusted infrequently and that rate changes usually occur in a sequence of consecutive small steps.

One possibility is to set interest rates according to a function of observed macroeconomic data. The Taylor (1993) rule is the canonical example of such an interest rate rule. An interest rate rule of one kind or another is now an integral part of most quantitative monetary models. Nevertheless, practitioners of monetary policy, for example, Bernanke (2015), frequently argue that the guidance provided by interest rate rules is limited due to, among others, the difficulties in assessing the theoretically appropriate conditioning variables, such as output gaps and natural real rates, in real time.

Price level indeterminacy under pure interest rate rules First, consider a monetary policy that would just specify a sequence of nominal interest rates. For a long time, such a policy has been viewed as perilous, as it leaves the price level indeterminate. To see why, consider a pure interest-rate peg in the context of the general-equilibrium cash-in-advance model from Section 17.3.2. In particular, assume a path for the nominal interest rate for which $i_t > 0$. As in our discussion of hyperinflation, we specialize utility to be linear in leisure; in particular, let period utility be $u(c) + l$ and let labor productivity z be equal to one. Then recall that the equilibrium conditions, which we only studied in a steady-state version in section 17.3.2, will read (in the order stated in that section) $c_t \leq p_{m,t}M_t$, $u'(c_t) = \lambda_t + \mu_t$, $1 = \lambda_t$, $q_t\lambda_t = \beta\lambda_{t+1}$, and $\lambda_t p_{m,t}/p_{m,t+1} = \beta(\lambda_{t+1} + \mu_{t+1})$. From this follows that $q_t = \beta$ and that $p_{m,t}/p_{m,t+1} = \beta(1 + i_t)$, using the Fisher equation, for all $t \geq 0$. Thus, the interest-rate peg determines inflation. This also means, using the last equilibrium condition, that $p_{m,t}/p_{m,t+1} = \beta u'(c_{t+1})$, thus, pinning down c_{t+1} for all $t \geq 0$. Thus, conditional on the initial price of money, $p_{m,0}$, the peg delivers a unique deterministic equilibrium from the second period on.

However, $p_{m,0}$ is not determined. Suppose that the CIA constraint in the initial period does not bind, that is, the initial CIA multiplier is zero, $\mu_0 = 0$, and consumption is determined by $u'(c_0^*) = 1$. For the given initial money stock M_0 any positive initial price of money $p_{m,0} > p_{m,0}^* = c_0^*/M_0$ will then satisfy the CIA constraint and represent an equilibrium. Alternatively, if $p_{m,0} \leq p_{m,0}^*$ then the initial CIA constraint binds, and the initial price together with the CIA constraint determines initial consumption, c_0, and the FOC for consumption determines the CIA multiplier, μ_0. So, any initial positive price of money indexes an equilibrium, and there is real, and not just nominal, indeterminacy.

Interest rate rules with a target Now consider amending monetary policy with an *interest rate rule*: a function relating the set interest rate to the price level (or, more commonly in practice, to the inflation rate, as in the Taylor rule)

$$i_t = \phi\left(\frac{P_t}{\widetilde{P}_t}\right). \tag{17.39}$$

Here, $\widetilde{P}_t$ is a target price level, which can change over time. The idea is that, under appropriate restrictions on the function ϕ, the basic price level indeterminacy disappears.[13]

How does the addition of the interest rate rule affect the determination of the price level? Let us go through the case of money in the utility function. Suppose, for a moment, that the real side of the economy is not affected by nominal variables. That would mean that the path for r_t can be taken as given. To make matters even simpler, suppose that r_t is constant: $r_t = r$ for all t. The Fisher equation then reads $1 + i_t = (1 + r)(P_{t+1}/P_t)$. It follows that if the policy attains its target price level in every period, the target interest rate is $\tilde{i} = \phi(1)$, and the target price level must change at a constant target inflation rate, $(1 + \tilde{i})/(1 + r) = \widetilde{P}'/\widetilde{P} = 1 + \widetilde{\pi}$.

[13] Note that from now on the discussion will be in terms of the price level, P, and no longer in terms of the price of money, p_m.

Combining the Fisher equation with the interest rate rule we can then write the evolution of the price level deviation from target as

$$\frac{P_{t+1}}{\widetilde{P}_{t+1}} = \frac{P_t(1+i_t)/(1+r)}{(1+\widetilde{\pi})\widetilde{P}_t} = \frac{1+\phi\left(P_t/\widetilde{P}_t\right)}{1+\widetilde{i}}\frac{P_t}{\widetilde{P}_t}. \tag{17.40}$$

Equation (17.40) thus defines a first-order difference equation for the nominal price level relative to its target, without an initial condition. Its local dynamics around a steady state of 1 are straightforwardly determined by the use of a first-order Taylor approximation: $P_{t+1}/\widetilde{P}_{t+1} - 1 \approx \left[1+\phi'(1)/(1+\widetilde{i})\right]\left(P_t/\widetilde{P}_t - 1\right)$. If $\phi'(1) > 0$, the local dynamics are determinate: there is a unique bounded solution with $P_t = \widetilde{P}_t$ for all t.[14] If, on the other hand, $\phi'(1) < 0$, then there is indeterminacy: a continuum of paths are consistent with convergence to the given steady state.[15] In conclusion, we see that if the central bank uses a rule that raises the nominal interest rate in response to an increase of the price level relative to its target, then the indeterminacy of nominal prices is no longer a problem. The corresponding demand for nominal money then follows from the money demand equation (17.35).

The discussion here has taken as given an exogenous path for r_t that, moreover, is constant. The arguments extend to an exogenous path for r_t that converges to a constant. How restrictive is the assumption that the path for r_t is exogenous? We know that the Euler equation for the real rate when money is in the utility function from equations (17.31) and (17.33) will, in general, involve nominal variables and hence make the analysis more involved. A full equilibrium treatment would require specification of the production side, and the equilibrium conditions would then need to be solved for. One possibility is that we again assume that production is linear in labor, $c = z(1-l)$, and that the government does not consume. Then the consumer's first-order condition for leisure, $zu_1(c_t, M_{t+1}/P_t) = v'(l_t)$, can be used directly to solve for time-independent $c_t = c$ and $l_t = l$ if we also assume that u is separable in consumption and real money balances. Hence, we can obtain $r_t = r$.[16] For the more general case of non-separability, one would need to examine the joint system of prices and real money holdings. It is of course possible to do so and to establish joint conditions on ϕ and u such that the equilibrium is determinate. The details are not important, so we omit them here. Nevertheless, dealing with the general case is relevant since a utility function where money enters separately from consumption is hard to motivate: after all, money's role is meant to be tied to the purchase of consumption goods.

Paying interest on money Recall Friedman's proposal: to pay interest on money. It is indeed conceivable for a central bank to pay interest on money, and it is even a policy in practical use, because many central banks pay interest on reserves, which are commercial bank accounts with a central bank and are part of what is considered money. Let i_m denote the interest paid on money. If $i_m = i$ at all times, money is equivalent to bonds in financial

[14] Here we focus on determinacy of bounded paths for the economy. Unbounded equilibria would need to be ruled out by other arguments.

[15] The knife-edge case $\phi'(1) = 0$ leads to indeterminacy as well: a continuum of non-diverging price ratios $P_t/\widetilde{P}_t$ are possible (constant) solutions to the equilibrium equations.

[16] This argument also works if z depends on time but is converging to a constant, in which case r_t converges to a constant.

terms for the household and no reduced-form approach is necessary for obtaining a value of money. If $i_m < i$, money is dominated in return by bonds and a reduced-form demand would need to be added for money to have value in equilibrium. Now, we can add interest on money in the budget constraint (17.11) with total real wealth $\hat{a}$ and the static opportunity costs of holding bonds or money, and the real opportunity cost of holding each unit of money becomes $(i - i_m)/(1 + i)$ multiplying M'/P. Using an $i_m \in (0, i)$ as an additional policy variable would not add conceptually to the rest of the analysis in this chapter, which is why our benchmark maintains $i_m = 0$.

The cashless limit

An important part of the New Keynesian model, to which Chapter 18 is devoted, deals with price level determination in the absence of a demand for money: money balances are typically omitted from that model. The price level is still nominal—set in units of currency—and serves a different role: it is the unit in which prices are assumed to be sticky. Price stickiness is not the subject of the present chapter, however; suffice it to say that in the New Keynesian model, a firm's price *in dollar terms* cannot be changed freely. But how is it possible to have nominal variables in the model without having money? We now discuss how Woodford (2003a) motivates this approach.

So consider again the model with money in the utility function. We just saw that a framework in which u is additively separable in consumption and real money balances is consistent with price-level determinacy under an interest rate rule, which is also the standard assumption about monetary policy made in the New Keynesian model. Moreover, with separable preferences, money balances do not matter for the determination of either real variables or the price level, but they are simply given *residually* from the money demand relationship (17.35). This economy is not cashless but it is consistent with using nominal variables without having to mention money balances. However, we also pointed out that additive separability is a very special, and arguably unrealistic, case.

Woodford considers the limiting *cashless* case for the more general model without additive separability. Thus money matters for real allocations and for price-level determination, but it matters less and less. For this approach we assume that the "relevance" of real money balances in the utility function (in comparison to consumption) can be represented by a parameter, ω, that can be taken to zero. Is it possible that for this limit in equilibrium (i) real money balances go to zero, while (ii) the price level remains finite and determinate and the remaining real variables can be pinned down as well? For this case consumers would demand less and less money from the central bank (in nominal terms), since they care less and less about money as ω approaches zero. Let us look into this possibility here.

Thus, consider the Euler equation for nominal bonds,

$$1 + i_t = \beta^{-1} \frac{u_{1,t}(c_t, M_{t+1}/P_t)}{u_{1,t+1}(c_{t+1}, M_{t+2}/P_{t+1})} \frac{P_{t+1}}{P_t},$$

which combines the Euler equation for real bonds, equations (17.31) and (17.33), with the Fisher equation, together with the money demand equation (17.35),

$$\frac{u_{2,t}(c_t, M_{t+1}/P_t)}{u_{1,t}(c_t, M_{t+1}/P_t)} = \frac{i_t}{1 + i_t}.$$

These two equations are interdependent but let us consider the first equation only in a limiting case. Let us again focus on an interest rate rule with a constant inflation target, $1+\widetilde{\pi}=\widetilde{P}_{t+1}/\widetilde{P}_t$, and implicit interest rate target, $1+\widetilde{i}=1+\phi(1)=(1+r)(1+\widetilde{\pi})$. With constant consumption in steady state, we have $1+r=1/\beta$. The idea is now to examine local determinacy around steady states, with successively smaller ω's, i.e., with lower and lower values for real money balances in steady state, with a particular focus on the limiting case of zero. We will define the local dynamics in logarithms; the interpretation is thus one of how percentage changes in variables relate to each other, which is a robust notion also when some variables are close to zero. We thus define

$$\widehat{i}_t \equiv \log\frac{1+i_t}{1+\widetilde{i}}, \quad \widehat{c}_t \equiv \log\frac{c_t}{\bar{c}}, \quad \widehat{m}_{1,t} \equiv \log\frac{M_{t+1}/P_t}{\bar{m}_1}, \quad \text{and} \quad \widehat{\pi}_{t+1} \equiv \log\frac{P_{t+1}/P_t}{1+\bar{\pi}}$$

where bars denote the steady states of variables associated with the policy implied interest rate $\widetilde{i}$. To make clear that our definition of real balances purchased at time t, $m_{1,t}=M_{t+1}/P_t$, differs from our previous definition of beginning of period real balances, $m_t=M_t/P_t$, we have added the subscript one.

We obtain the first-order Taylor expansion of the Euler equation,

$$\widehat{i}_t=\eta_{u_1,c}\left(\widehat{c}_{t+1}-\widehat{c}_t\right)+\eta_{u_1,m_1}\left(\widehat{m}_{1,t+1}-\widehat{m}_{1,t}\right)+\widehat{\pi}_{t+1},$$

where

$$\eta_{u_1,c}\equiv-\frac{\partial u_1}{\partial c}\frac{c}{u_1}$$

and

$$\eta_{u_1,m_1}\equiv-\frac{\partial u_1}{\partial m_1}\frac{m_1}{u_1}.$$

The idea is now to parameterize the utility function with an ω such that, when ω approaches zero, the associated steady-state $\bar{m}_1$ value goes to zero, η_{u_1,m_1} goes to zero, and $\eta_{u_1,c}$ goes to a finite value. If so, the dynamics implied by the linearized Euler equation do not involve money at all, so together with the interest rate rule, it defines a complete system.

A complete characterization of the class of utility functions satisfying these requirements is beyond the scope of the treatment here. An example, however, can be provided. So suppose that u is an increasing, strictly concave function, f, of a constant returns-to-scale CES index, h, in the two arguments c and m_1

$$h(c,m_1)=\left[(1-\omega)c^{\rho}+\omega m_1^{\rho}\right]^{1/\rho} \text{ with } \rho<1.$$

Here our key parameter appears: ω is the weight on real balances. With this formulation, the money demand equation, $u_2(c,m_1)/u_1(c,m_1)=h_{m_1}/h_c=i/(1+i)$, becomes

$$\frac{M_{t+1}}{P_t}=c_t\left(\frac{1-\omega}{\omega}\cdot\frac{i_t}{1+i_t}\right)^{\frac{1}{\rho-1}}. \tag{17.41}$$

This implies a money demand which has unitary elasticity with respect to consumption and a constant (negative) elasticity with respect to the cost of holding money, $i/(1+i)$. Since $\rho<1$ the demand for real balances is a decreasing function of the opportunity cost of holding money. These features hold regardless of the value of ω: if $\omega \to 0$, then $M_{t+1}/P_t \to 0$ but the elasticities remain unchanged. Thus, if P_t is targeted to equal $\widetilde{P}_t$, with the aid of the interest rate rule, it must be that $M_{t+1} \to 0$.[17]

It remains to be shown that when ω approaches zero, η_{u_1,m_1} goes to zero while $\eta_{u_1,c}$ goes to a positive constant. It is straightforward, but somewhat tedious, to show this to be true under the functional assumptions given.[18]

Notice that the interest-rate rule is critical for the logic of the cashless limit to go through: the interest rate is fixed in a relation to prices such that, as ω goes to zero, M/P goes to zero through M going to zero. If monetary policy was instead governed by a money supply rule—the simplest form of which is to have M constant over time—then P would go to infinity (and p_m to zero) as ω and M/P went to zero, and the New Keynesian model could not be built on a core where the price level is infinity.

Monetary-fiscal interactions

In this section we look more carefully at the government's budget constraint and how it interacts with monetary policy. This will allow us to touch on a number of conceptual issues. Throughout, we maintain a consolidated view of the government sector, i.e., the fiscal authority plus the central bank. We do not model their behavior but rather discuss the set of policies they could undertake. An alternative would be to formulate a game between two authorities with different objective functions, and a private sector responding to policy, but there is no established approach to that in the literature.

Financing a given path of primary deficits In real terms, the flow budget constraint of the government reads

$$g_t + (1 + i_{t-1})\frac{B_t}{P_t} = \tau_t + \frac{B_{t+1}}{P_t} + \frac{M_{t+1} - M_t}{P_t}. \tag{17.42}$$

Here, the left-hand side is government spending—real purchases of goods plus debt repayment, including interest—and the right-hand side describes how spending is financed: via taxes, new borrowing, or money printing, also known as seigniorage. This budget constraint consolidates the budgets of the fiscal and monetary authorities; in particular, B denotes debt to the public, and any debts between the fiscal and monetary authorities net to zero. In

[17] For a more general utility function, we can rewrite the implicit money demand equation as

$$\frac{i}{1+i}\frac{m_1}{c} = \frac{u_2(c, m_1)}{u_1(c, m_1)}\frac{m_1}{c} = \frac{u_{1,2}m_1/u_1}{u_{2,1}c/u_2} = \frac{\eta_{u_1,m_1}}{\eta_{u_2,c}}$$

since second derivatives (represented by $u_{i,j}$) are symmetric, $u_{1,2} = u_{2,1}$. A more general condition that delivers $m_1/c \to 0$ under a constant interest rate is therefore that $\eta_{u_1,m_1}/\eta_{u_2,c} \to 0$ as $m_1/p \to 0$, i.e., that the ratio of cross-elasticities of marginal utility goes to zero.

[18] Note that $u_{1,1}c/u_1 = (f''c/f')h_1 + h_{1,1}c/h_1$ since $u_1 = f'h_1$. It is easy to see, by taking successive derivatives of the CES function h, that in the limit as $\omega \to 0$, the second term $h_{1,1}c/h_1 \to 0$. However, the first term becomes $f''(c)c/f'(c)$, which is strictly negative; hence, $\eta_{u_1,c}$ is strictly positive. Turning to $u_{1,2}m_1/u_1$, we obtain $(f''/f')h_{2,2} + h_{1,2}m_1/h_1$. As $\omega \to 0$, both these terms go to zero: the h function has the property that both $h_{m_1m_1}$ and $h_{1,2}m_1/h_1$ go to zero. Hence η_{u_1,m_1} goes to zero.

equilibrium, the present value of the budget constraint reads

$$\sum_{t=0}^{\infty} q_{0,t}\,(g_t - \tau_t) + (1+i_{-1})\frac{B_0}{P_0} = \sum_{t=0}^{\infty} q_{0,t}\frac{M_{t+1}-M_t}{P_t}, \tag{17.43}$$

where discounting, $q_{0,t} = \prod_{s=0}^{t} q_s = 1/\prod_{s=0}^{t}(1+r_s)$, uses the Fisher equation.[19] The left-hand side of this equation is the present value of the primary deficit, $g-\tau$, plus the initial debt (including interest), and the right-hand side indicates the total remaining financing needs that will have to be covered by money printing. In this obvious sense, the central bank's operations play a role in the consolidated budget.

There is also an alternative description of the consolidated constraint, which we can obtain by defining $D_t \equiv (1+i_{t-1})B_t+M_t$ as the total liabilities of the consolidated government at time t. Here, M is not a liability in the usual sense of something having to be paid back, but rather it is an item on the central bank's balance sheet.[20] Using this definition, the flow budget constraint can be written (along the lines of the description of the household's constraint, equation (17.11)) as

$$d_t = \tau_t - g_t + \frac{M_{t+1}}{P_t}\frac{i_t}{1+i_t} + \frac{1}{1+r_t}\cdot d_{t+1}. \tag{17.44}$$

Thus, the given initial liabilities $d = D/P$, which can be seen as a present value, equal the current primary surplus, $\tau - g$, plus the seigniorage revenue from the current presence of money in D, plus the present value of the liabilities left for next period, d'. This notion of seigniorage is more narrow and it represents a form of arbitrage that the government carries out: to the extent the private sector values money for non-financial reasons, the government saves on its financing costs by substituting non-interest-bearing money, M, for interest-bearing debt, B, while maintaining the same value for total liabilities, D. The interest savings, per unit of money, is $i_t/(1+i_t)$, and the total savings are larger, the larger the real balances share of the liabilities. We can also consider this form of seigniorage as that accruing for a fixed path of D/P.

Equilibrium seigniorage in the long run and fiscal dominance The previous discussion merely describes government variables, assuming that in equilibrium money and bonds are valued by private agents in the amounts specified. The value of money to the private sector has been the subject of previous sections. In the overlapping-generations setting, money serves as a mere storage instrument and only has value ($P < \infty$) if nominal interest rates are zero. Let us instead adopt the reduced-form liquidity perspective where we can

[19] One might think that the government's present value budget constraint represents the forward solution of the flow budget constraint (17.42), subject to a no-Ponzi game condition, analogous to the household's budget constraint in Chapter 4, Section 4.3.1. There is, however, no optimization problem associated with the government, and thus there is no reason to impose an no-Ponzi-game condition on the government. Rather, we derive the government's present value budget constraint from the household's present value budget constraint by imposing market clearing. In this sense, the present value budget constraint (17.43) represents both a constraint and an equilibrium condition.

[20] In the distant past, when some central banks promised that each unit of money could be exchanged for a fixed number of units of gold, the liability was more apparent.

derive a static money demand equation as $M_{t+1}/P_t = c_t f(i_t)$, with c being private-sector consumption and f a strictly decreasing function.[21] Thus, taking c_t as given, the arbitrage-based seigniorage expressed in (17.44) is proportional to $f(i)i/(1+i)$. This function has a decreasing part, $f(i)$, multiplying an increasing part, $i/(1+i)$, that starts at 0 and is bounded above by 1. So, under rather weak assumptions on f, the product expression will have a maximum and describe a "Laffer curve." As the nominal interest rate i increases above zero, seigniorage revenues are obtained, but at some point, further increases in i will lead to declining revenues.

Let us now consider a stationary, long-run equilibrium: taxes and spending are constant, $\bar{\tau}$ and $\bar{g}$, as is consumption, $\bar{c}$, inflation, $\bar{\pi} = P'/P$, the real and nominal interest rate, $\bar{q} = 1/(1+\bar{r})$ and $\bar{i}$, and the real liability and money demand levels, $\bar{d}$ and $\bar{m}$. Then, the government budget constraint (17.44) will read

$$\bar{g} + \frac{\bar{r}}{1+\bar{r}}\bar{d} = \bar{\tau} + \bar{c}f(\bar{i})\frac{\bar{i}}{1+\bar{i}}.$$

Many of the models in the literature have exact, or approximate, separation between purely monetary and real phenomena in the long run. With that motivation, let us consider $\bar{g}$, $\bar{r}$, and $\bar{c}$ to be given here. Then an increase in government liabilities (be it through an increase in money or bonds) must cause an adjustment in the nominal interest rate. If, in addition, the economy is on the "benign" side of the Laffer curve, then an increased $\bar{d}$ will raise $\bar{i}$. Because $\bar{r}$ is given, the Fisher equation implies that inflation will need to rise.

The long-run version of the government's budget constraint allows us to touch on the well-known Sargent and Wallace (1985) piece "Some Unpleasant Monetarist Arithmetic." In that paper, the authors argue that if the central bank at any point wishes to fight inflation through an open-market operation where the money stock is decreased and the stock of debt is increased (borrowing to buy and withdraw money from the economy), then because debt pays interest, $\bar{d}$ will rise. So in the long run, $\bar{d}$ will be higher (unless the short-run action is reversed), which will increase $\bar{i}$ and, therefore, inflation. Thus, the arithmetic of the consolidated government's budget makes fighting inflation difficult if the fiscal authority does not collaborate by increasing its primary surplus. Sargent and Wallace framed their argument as one with a "fiscal dominance regime." That is, the fiscal authority independently commits to primary surpluses and leaves it to the monetary authority to adjust as necessary to satisfy the government's intertemporal budget constraint. In such a situation, even a monetarist central bank may not be able to fight inflation. Their argument can be made without restricting the analysis to a steady state, but we omit the details here.

The irrelevance of open-market operations Wallace (1981b) considers a situation where open-market operations, that is, changes in the central bank's balance sheet involving exchanges of money for outstanding government bonds, do not affect the economy at all. This occurs in a special situation where money functions as a store of value and is not dominated in return by government bonds. The consolidated budget setting here allows us to explain the irrelevance; the original argument in Wallace (1981) instead used the overlapping-generations model of money to derive the result.

[21] The unitary elasticity in c obtains under special functional-form assumptions only but is not key to the argument here.

Recall again the budget constraint (17.44) and suppose now that $i_t = 0$ for all t. The budget then simply reads

$$\frac{M_t + B_t}{P_t} = \tau_t - g_t + \frac{1}{r_t}\frac{M_{t+1} + B_{t+1}}{P_{t+1}}.$$

It is not clear from this budget constraint whether (and why) money has value, but the maintained assumption is that the private sector regards M and B as perfect substitutes. In addition, it is not clear how many equilibria may exist. However, the following can be said: if an equilibrium exists, in the form of a set of sequences $\{M_t, B_t, \tau_t, g_t, r_t\}_{t=0}^{\infty}$ along with other variables (consumption, output, relative prices, etc.), then any $\left\{\widehat{M}_t, \widehat{B}_t, \tau_t, g_t, r_t\right\}_{t=0}^{\infty}$ with other variables maintaining their values, is also an equilibrium so long as $\widehat{M}_t + \widehat{B}_t = M_t + B_t$ for all t. The proof is trivial: the government's budget constraint is still met and the private sector's situation is entirely unchanged, so if the original sequences constitute an equilibrium, so does the proposed alternative.

All in all, the only truly central assumption behind the irrelevance proposition is that money and bonds be perfect substitutes. This assumption might make you think that the proposition itself is irrelevant: in reality, bonds pay interest and money does not! However, the recent long period of zero interest rates again made the proposition relevant: it suggests that *quantitative easing*, QE, is irrelevant at the zero lower bound. For QE to have real impact, bonds and money have to differ for some reason. Most discussions of QE focus on differences in duration—central banks exchange money for long-term bonds. The irrelevance proposition thus forces us to think more about how money and bonds differ.

Indeterminacy issues: fiscal back stops and the fiscal theory of the price level We have seen, in the various discussions of different models of money, that monetary equilibria may not be unique. Moreover, the number of equilibria may depend on the monetary policy assumed. We now consider two examples. In the first example, a "fiscal backstop" eliminates all equilibria for which money loses value in the limit. In the second example, illustrating the *fiscal theory of the price level*, a policy that fixes the present value of government net-revenues eliminates a continuum of equilibria that start away from the stationary equilibrium but eventually converge to it. In either case, we are left with the unique stationary equilibrium.

So, first, recall the hyperinflation example above with a continuum of equilibria indexed by the initial price level P_0, where the equilibrium price path increases without bounds for any initial $P_0 > \bar{P}_0$ in such a way that money loses value entirely in the limit. The fiscal-monetary policy specification for this example was constant money growth $M_t = \gamma M_{t-1}$ implemented through lump-sum transfers, that is, negative taxes. A simple fiscal backstop that eliminates the equilibria with price level growth exceeding money growth is one where the government promises to buy an unlimited amount of money if, at any time, the price level exceeds a critical value, $\widetilde{P}_t = \widetilde{P}_0\gamma^t$, with the purchases financed by lump-sum taxes; see Wallace (1981a) and Obstfeld and Rogoff (1983). The critical value grows at the rate γ, since in the stationary equilibrium with constant real balances the price level grows at the money growth rate. If the market considers this policy credible, the hyperinflationary equilibria cease to exist. Thus, fiscal policy supports a monetary equilibrium through an off-equilibrium action.[22]

Now consider another example where fiscal policy is used to rule out indeterminacy of the equilibrium path. In particular, consider the pure interest-rate peg in the context of the general-equilibrium cash-in-advance model. For this policy the initial price level, P_0, is indeterminate and indexes all perfect foresight equilibria. Furthermore, the interest rate peg is supported by a path for the nominal money stock, M_{t+1}. We now describe a policy that eliminates equilibrium indeterminacy. We will consider a specific policy for simplicity, but the argument is general in nature: it only relies on the government's budget constraint and an assumption about what variables adjust to make sure it is satisfied. Hence, the argument applies for a general class of models displaying indeterminacy. We will consider a simple rule for lump-sum taxes

$$\tau_t - \tilde{\tau} = -\frac{M_{t+1} - M_t}{P_t} \tag{17.45}$$

for a given $\tilde{\tau}$ and fixed spending $g_t = \tilde{g}$. Here, lump-sum taxes decline one-for-one with the magnitude of the real money stock change necessary to support the interest rate peg on the equilibrium path. Thus, if seigniorage were to decline, so that the right-hand side would rise, the lump-sum tax would go up to compensate for the seigniorage shortfall. Substituting the tax rule into the government's budget constraint (17.43) yields

$$\frac{(1+i_{-1})B_0}{P_0} = \sum_{t=0}^{\infty} \beta^t \left(\tau_t - g_t + \frac{M_{t+1} - M_t}{P_t} \right) = \sum_{t=0}^{\infty} \beta^t \left(\tilde{\tau} - \tilde{g} \right) = \frac{\tilde{\tau} - \tilde{g}}{1-\beta} \tag{17.46}$$

Here, the only endogenous variable is the price level P_0, and the promise of future fiscal adjustments thus implies a unique solution for the price level. The *fiscal theory of the price level* refers to the notion that commitment to a fiscal rule for the indefinite future determines the present value of primary surpluses, which forces the price level to adjust so that the initial real debt is covered by the present value of future surpluses. The argument of course requires that $B_0 > 0$, but the other details of our example are not important. One can view government debt here as an asset: a (forward-looking) claim to future payments in the form of primary surpluses.

Compare the outcome for this fiscal rule with the outcome for an interest rate peg we studied above when the interest rate is determined by a state contingent rule, equation (17.39). There we have shown that if the interest rate rule is sufficiently responsive to deviations of the price level from a target the equilibrium is also unique. But if the price level is unique, how can we be sure that it is consistent with the price level as determined by the government's present value budget constraint? Here it becomes important that the present value government budget constraint represents an equilibrium outcome. Therefore if the nominal interest rate rule (17.39) results in a unique equilibrium with associated paths for the price level and money stock then fiscal policy, lump sum taxes and spending, has to adjust such that the present value budget constraint holds. Again, the problem is related to the Sargent and Wallace (1985) piece "Some Unpleasant Monetarist Arithmetic": the price level being uniquely determined by the interest rate rule implicitly assumes a "monetary dominance" regime.

[22]Notice that this fiscal-monetary policy effectively amounts to a promise to give money value in the future, that is, money is "backed" (by goods, not gold).

There is debate among macroeconomists about the relevance of the fiscal theory of the price level. To some, it seems counterintuitive that price setters today—imagine a fruit seller—are modeled as needing to set the price so as to make sure that the value of the government's debt becomes equal to a certain value. But note that we are talking about equilibrium prices in a dynamic competitive equilibrium, and in general, we study the properties of an equilibrium and do not ask how an equilibrium comes about.

17.4 Multiple currencies

What determines the exchange rate between two currencies, such as the dollar and the euro? This is a major question in international macroeconomics. Undergraduate textbook treatments refer to certain "parity" conditions—purchasing power parity or (covered or uncovered) interest rate parity—as guidance. Here, the purpose is to discuss exchange rate determination from the perspective of the models considered so far. In so doing, we will also more generally touch on implications for determining the value of other money-like assets, such as crypto-currency.

17.4.1 Money as a store of value: Kareken-Wallace exchange rate indeterminacy

What if we introduced more than one type of money (or currency) into models where money's only role is that of a store of value, such as the overlapping-generations model? The main discussion here will be in the context of one country only. One reason for this is that it simplifies the exposition. Another is that consumer heterogeneity or multiple types of goods is not central to the argument. We will also consider the possibility that currency stocks grow at different rates, mimicking differences in monetary policies across countries. The discussion follows Kareken and Wallace (1981).

So, consider an overlapping-generations model with one good, one representative consumer per cohort, and two currencies, a and b, for the consumer to invest in. The prices of the two currencies in terms of the consumption good are p_a and p_b. The consumer born at t thus faces budget constraints

$$c_y + p_{a,t}M_a' + p_{b,t}M_b' = \omega_y \quad \text{and} \quad c_o = \omega_o + p_{a,t+1}M_a' + p_{b,t+1}M_b',$$

and the non-negativity constraints, $M_a' \geq 0$ and $M_b' \geq 0$. The choice is over (c_y, M_a', M_b', c_o) and let us for simplicity use $u(c_y, c_o) = \log c_y + \log c_o$ as the objective function to maximize. Here, the consumer faces four prices but what matters for decision-making are the real returns $p_{a,t+1}/p_{a,t}$ and $p_{b,t+1}/p_{b,t}$. Because short-selling is not possible, the consumer cannot conduct arbitrage based on rate of return differences between the currencies. Thus, if at any point in time t one currency has a lower return than the other currency, no consumer will hold the low-return currency: its price will be zero. It then follows that the price would be zero also before that date since no consumer would buy the currency for a positive price and sell it later for a zero price. Similarly, the price of this currency after the date t would also have to be zero since otherwise, markets would not clear in the period before the price becomes positive: consumers would demand an infinite amount of currency at that point. Hence possible equilibria have the structure that at all times, either (i) both currencies are valued; (ii) only one of the currencies is valued; or (iii) no currency is valued. As the discussion in Section 17.2 should make clear, case (iii) will apply if $\omega_y \leq \omega_o$. If $\omega_y > \omega_o$, however, (i) and (ii) are possible. But case (ii) is subsumed in our previous analysis. If, for example, future agents do not value currency a, present agents will not either, and hence $p_{a,t} = 0$ at all times and we are back to the overlapping-generations economy with one currency. So instead consider case (i) and let e_t be the *nominal exchange rate* between the currencies, $e_t \equiv P_{b,t}/P_{a,t} = p_{a,t}/p_{b,t}$. The exchange rate measures how many units of currency b need to be given up to obtain one unit of currency a: if e is above 1, one unit of currency a is more valuable than one unit of currency b. Clearly, from $p_{a,t+1}/p_{a,t} = p_{b,t+1}/p_{b,t}$ at all times, it follows that e_t must be constant over time: $e_t = e$. Moreover, we can define the total money holdings, in currency b units, as $M' \equiv eM'_a + M'_b$ and $p_b M'$ as its real value. Thus, the consumer's budget constraints become

$$c_y + p_{b,t}M' = \omega_y \quad \text{and} \quad c_o = \omega_o + p_{b,t+1}M',$$

and, just as in Section 17.2, demand for total money by the consumer born at t satisfies the equation $p_{b,t}M_{t+1} = \max\{(\omega_y - \omega_o p_{b,t}/p_{b,t+1})/2, 0\}$. To close the model, we need to specify money supplies. Suppose they are constant over time: for all t, $M_{a,t} = M_a$ and $M_{b,t} = M_b$. Thus, focusing on equilibria where money has value, $M = eM_a + M_b$ and the equilibrium is determined by

$$p_{b,t}M = \frac{1}{2}\left(\omega_y - \frac{\omega_o}{p_{b,t+1}/p_{b,t}}\right)$$

holding at all times. As in the one-currency model, this defines a set of equilibrium sequences for $p_{b,t}M$: one is a steady state, where $p_{b,t}M$ is constant and equal to $(\omega_y - \omega_o)/2$; but there is also a continuum of other paths with $p_{b,t}M$ converging to zero over time. The key observation here, however, is that there is no other equilibrium condition, and hence e is not determined. To be more concrete: select an arbitrary $e \in (0, \infty)$. This will define $M = eM_a + M_b$, since (M_a, M_b) are given. Then $p_{b,t}$ follows from knowing $p_{b,t}M$ in the given equilibrium. This allows us to find $p_{a,t}$: it equals $ep_{b,t}$ at all times. Since the consumer's demands for individual currencies are not pinned down—there is complete indifference, since the currencies give identical returns—we can then set $M_{a,t+1} = M_a$ and $M_{b,t+1} = M_b$ at all times, since their value sum is then equal to their chosen total amount of saving $(eM_a + M_b)p_{b,t}$.

Suppose now that the currency stocks change over time because the government provides lump-sum transfers of the two currencies to the agents when old. We can still, for an arbitrary e, define a total money supply as $M_t = eM_{a,t} + M_{b,t}$, which will now in general depend on time. Incorporating the lump-sum transfers and after some algebra we derive a modified demand for money that again defines a difference equation for real balances

$$p_{b,t}M_t = \frac{1}{2}\left\{\omega_y - \frac{\omega_o + p_{b,t+1}(\gamma_t - 1)M_t}{p_{b,t+1}/p_{b,t}}\right\},$$

where $\gamma_t = M_{t+1}/M_t$. A solution will be nonstationary because γ_t appears in the equation. Consider the simple case where each money stock grows at a constant gross rate, γ_a and γ_b, and without loss of generality, suppose $\gamma_a \geq \gamma_b$. Then γ_t will converge to γ_a. One can show that an equilibrium exists where total real balances converge to a positive constant with the limit being $p_b M = (\omega_y - \gamma_a \omega_o) / (1 + \gamma_a)$.[23] Thus faster-growing money will dictate the long-run real value of the total money stock, and it will comprise all of the money stock, $eM_a/M = 1$. This can be interpreted as a version of Gresham's law: "bad money drives out good money," where "bad" refers to a faster-growing stock. This statement, of course, is conditioned on an equilibrium of type (i); another equilibrium is always that where $e = 0$, in which the value of the total money stock will converge to $p_b M = (\omega_y - \gamma_b \omega_o) / (1 + \gamma_b)$, which is higher.

17.4.2 Dynastic models with a reduced-form liquidity demand

With a reduced-form liquidity demand, a key question immediately arises: how should the two currencies appear (in cash-in-advance constraints, in a transactions-cost technology, or in utility)? In an early paper, Lucas (1982) considers a two-country model where there are two traded goods and consumers in both countries value both goods. Country 1 consumers, however, are only endowed with goods of type 1 and country 2 consumers are only endowed with goods of type 2. He, then, assumes that good 1 is subject to a cash-in-advance constraint involving only country 1 money, and similarly for good 2: you need country 2 money to buy it. Therefore, all consumers need both types of currency. The model gives a uniquely determined exchange rate: the value of country 1 money is tied, via the cash-in-advance constraint, to the total demand for good 1, which is real. Thus, the exchange rate is directly tied to the relative demands for the two consumption goods (and to the relative money stocks).

A similar result to Lucas's can be obtained with any of the other models where money has reduced-form liquidity demand. One can, for example, assume that foreign money has some (quantitatively limited) value to one's utility. Monetary policies will matter for exchange rates since the real value of money is pinned down by its reduced-form real role, as well as its financial return. If a country increases its money stock at a slower rate than other countries, then its exchange rate will appreciate over time, everything else equal.

Discussion: theory and data

Exchange rates fluctuate significantly over time and are typically described as random walks, that is, their movements are unpredictable. To what extent can the theories used here be used to understand these facts? The overlapping-generations model we studied above does not generate fluctuations: it predicts an indeterminate but constant exchange rate. However, for extensions of the model that introduce extrinsic uncertainty (i.e., non-fundamental random fluctuations, such as "sunspots") the indeterminacy then allows for unpredictable movements in exchange rates (martingales). Reduced-form liquidity models admit randomness in exchange rates to the extent that there is randomness in fundamentals, such as in money supplies or output.

Relatedly, practitioners (and basic undergraduate textbooks) often refer to *Purchasing Power Parity* (PPP) as a guide for understanding what the value of an exchange

[23] Since the proof is a bit complicated we leave it to the reader to consult the Appendix of Kareken and Wallace (1981).

rate should be: it should cost the same to buy a given set of (tradable) goods in one currency directly as it would if one swapped into another currency and bought the goods using that currency.[a] Domestic prices do not fluctuate nearly as much as do exchange rates, implying that PPP cannot hold at all points in time, even though it might hold on average over time. So to the extent one could identify a basket of goods that is available in two countries, couldn't this be a way to think about exchange rate determination? Clearly, in the models described above—the overlapping-generations model and the reduced-form liquidity models—PPP holds: there is free trade in goods and currencies. If purchasing power parity appears to be violated in the data, then in a strict sense it contradicts these theories. But the idea here would be to argue that the theories still hold on average over time, and hence they can be used to predict an upcoming adjustment in exchange rates in the direction of making PPP hold. However, PPP can be restored also by adjustments in the price levels: these are fundamentally endogenous. Prices may move slowly over time, but gradual movements would also allow us to move toward restoring PPP, thus not necessarily involving exchange rate adjustments. A similar condition is *interest rate parity*: the notion that investing in bonds in one country should give the same return as investing in bonds in another country. In particular, a dollar invested in U.S. bonds gives an interest of i_t between t and $t+1$; alternatively, the investor could buy euro at the time t exchange rate, invest in euro bonds paying $\tilde{i}_t$, and convert the proceeds back to dollars at the $t+1$ exchange rate. Again, in all the theories considered above, these two transactions would give the same return. Can apparent departures from this kind of parity be used to argue that the current exchange rate is too high or too low? No. Parity does not predict the exchange rate at t (conditional on knowing i_t and $\tilde{i}_t$), rather it predicts the exchange rate at $t+1$ relative to that at t. Hence, it cannot help us understand the level of an exchange rate.

Because of their large fluctuations and violation of parity conditions, exchange rate movements are challenging to understand with our basic theories. Exchange rate indeterminacy in the overlapping-generations model suggests a way of thinking about fluctuations. However, it does not account for why dollars are used in most transactions in the United States and euro in euro countries. Reduced-form models, with the stroke of a pen, allow for this feature. They also predict that if a country prints money at a faster rate than another country, and it experiences higher inflation as a result, then its exchange rate will depreciate. This feature is broadly in line with data. One interesting example is the Swiss franc, a currency that has experienced decades of consistent appreciation. Is this appreciation because Swiss inflation has been low in an international comparison? Maybe yes, but the appreciation is much stronger than what can be accounted for (using a PPP relationship) by its low inflation. Thus, the obvious fundamentals go some, but far from all of, the way toward understanding the facts.

[a] The so-called Big Mac index is another example of this idea, though a Big Mac can hardly be regarded as tradable.

17.4.3 Crypto-currency

Crypto-currency is a form of digital currency that can serve as a medium of exchange and a store of value. The circulation of alternative currencies has a long history, for example, paper money issued by commercial banks. Whether the digital nature of crypto-currency makes it special is not clear but it does have certain distinguishable properties: it is costless to "carry" and, typically, has a way of securing privacy in that your holdings are, for example, not directly observable to the government. Thus, crypto-currency may have a comparative advantage for criminal activities, especially since regulations limiting money laundering have become more and more potent in many countries. But crypto-currency can be given different formats too; in some versions, it is marketed as an asset simply with above-market return (for some time), and in others as a safe store of value (for example, the so-called stablecoins, whose value is tied to the dollar; see Azzimonti and Quadrini (2025)).

How can crypto-currency be understood given the above theories? The overlapping-generations model would say that it is just another currency that can, based on the expectations of the behavior of future consumers, potentially be used as a store of value. Thus, its value while indeterminate could be positive when introduced. And its exchange rate with standard currencies could move randomly. This theory does not rely on any of the specific properties of crypto-currency. The reduced-form models of liquidity could also accommodate more currencies. A cash-in-advance theory could allow other currencies as allowable means of payment for some or all of the goods. A money-in-the-utility function model could introduce another variable in utility, perhaps nested with standard money in a CES form. Of course, none of these approaches seem satisfactory since the results (in the form of the equilibrium value of crypto-currency) appear to follow very directly from the assumptions.

17.5 Missing assets

In the remaining sections of this chapter, we look at two more fundamental models of money in which valued money is not an assumption but is an outcome of the environment. As we have seen, in the overlapping-generations model, money can serve as a store of value under some restrictions on primitives, primarily involving utility functions and the time profile of endowments. In the present section, we look at an environment in which money as a store of value can (at least partially) replace some missing assets. We introduce this feature into a dynastic environment closely related to material covered earlier in the book: consumer heterogeneity and incomplete insurance against idiosyncratic shocks.

Consider the Huggett model studied in Chapter 11, with the tightest possible borrowing constraint: agents cannot borrow, $a' \geq 0$. In a stationary equilibrium agents solve

$$v_\omega(a) = \max_{a' \geq 0} u(a + \omega - qa') + \beta \int_{\omega'} v_{\omega'}(a') F(d\omega'|\omega),$$

where income ω follows a first-order Markov process. We obtain an autarky equilibrium where nobody borrows or lends. The agent valuing the bond the most is indifferent between borrowing and lending, and this agent's Euler-equation determines the interest rate; everybody else is borrowing constrained.

In the simplest, two-state case with nontrivial probabilities, we obtain

$$qu'(\omega_{hi}) = \beta \left[\pi_{hi|hi}u'(\omega_{hi}) + \pi_{lo|hi}u'(\omega_{lo})\right]$$

and hence the gross real interest rate is

$$\frac{1}{q} = \frac{1}{\beta} \cdot \frac{1}{\pi_{hi|hi} + \pi_{lo|hi}\frac{u'(\omega_{lo})}{u'(\omega_{hi})}} < \frac{1}{\beta}.$$

Now suppose $1/q < 1$, that is, we have a negative (net) real interest rate, like in the overlapping-generations case. Here as well, people would like to save (rather badly if $u'(\omega_{lo})/u'(\omega_{hi})$ is very high) but are lacking assets for it: intra-personal loans are not allowed because the asset cannot be held in negative amounts. That is, an asset is prevented from existing. Other assets are missing too—insurance claims written contingent on idiosyncratic endowment outcomes—but the key here is that a riskless asset is missing.

Now consider introducing fiat money in the economy: suppose there is a fixed stock of M nominal units. If the price level is constant, $P_t = P$, then money can play the role of a riskless asset, $m = a$, with gross real return one, $q = 1$, that helps consumers with high-endowment realizations save and smooth consumption.[24] What would the steady-state real value of this money be? The answer can be obtained by solving

$$v_\omega(a) = \max_{a' \geq 0} u(a + \omega - a') + \beta \int_{\omega'} v_{\omega'}(a')F(d\omega'|\omega);$$

with implied decision rule: $a' = g_\omega(a)$. Thus the total value of the money stock is simply the total savings in this economy

$$m = \int_{a \geq 0, \omega} g_\omega(a)\Gamma(d\omega, da),$$

where Γ is the stationary distribution implied by the decision rule and the distribution of endowment shocks. Given the fixed stock of nominal money, this expression then determines the price level. Of course, the role played by money could also be played by (real) government bonds.

An even simpler missing-asset model is a deterministic version of the model above where endowments alternate between high and low: for half of the population, endowments are high (low) in even (odd) periods, and for the other half of the population they are high (low) in odd (even) periods. If borrowing is not allowed (individual debt assets are ruled out), then the equilibrium would be autarky. But money could be introduced and would have value if the autarky interest rate in the original economy is below zero. This is essentially the environment of Townsend (1980)'s turnpike model of money. Like the overlapping-generations model, the missing-asset model has the attractive feature that it can account for valued money without simply assuming it, but it cannot explain why money continues to have value when being dominated in return. One might argue that even though money is not valued by assumption in these environments, money is valued because some other assets are excluded by assumption.

[24]Similar to the overlapping-generations model, one can also imagine non-stationary equilibria here, where money would lose value over time and be worthless in the limit.

17.6 Models of money as a medium of exchange

The cash-in-advance and transaction cost models described above share the feature that money is used to facilitate consumption and, in that sense, functions as a medium of exchange. However, these are not models that explain why money may be important in exchange. In undergraduate texts, the informal motivation for money as a medium of exchange is the *absence of double coincidence of wants*. That is, when buyers meet sellers they are rarely in a situation where a direct exchange of goods or services is mutually beneficial (without access to some form of public record-keeping technology). The model in Kiyotaki and Wright (1989) is the first that carefully spells out the friction that motivates this idea. We briefly describe a setting like theirs here (without equations). A modern version of their model is contained in Lagos and Wright (2005). In all these settings, trade is fundamentally *decentralized*: people don't (always, at least) trade in centralized markets. Also, markets are *anonymous*: there is no record-keeping. In this sense, they are similar to the search/matching model in Diamond (1982).

Kiyotaki and Wright's (1989) model We will describe the core setting and attempt to explain verbally how it works. There are three kinds of people, all deriving utility from the consumption of a good, but a consumer of type $i \in \{1, 2, 3\}$ only consumes good $i+1$, modulo 3 (i.e., $i + 1$ is interpreted as 1 when $i = 3$). We denote the utility benefit of consumption by u_i for consumer i. Moreover, there is production: upon consuming, a person of type i produces a good of type i and, hence, is also a producer. Hence, a person's identity i is the good she produces. All goods must be indivisible: production, when it occurs, always delivers one unit of a good. Each period, people meet pairwise. Thus, by construction, when people meet, there is never a double coincidence of consumption wants. For example, a person of type 1 can offer her produced good 1, but only a person of type 3 likes that good, and unfortunately that person produces good 3, which person 1 does not consume.

In this economy, goods can be stored. In particular, good i can be stored from t to $t + 1$ at a utility cost c_i that is additive (does not interact with consumption utility). Thus, it is conceivable that a producer of type i, when meeting someone, swaps the good just produced for another good with the idea of storing it and possibly swapping this good for good $i+1$ and consuming it in the future. The stored good would then function as a *medium of exchange*. Meetings are assumed to be random and a person of type i is equally likely to meet a person of type $i + 1$ and $i + 2$, modulo 3.

In each period of the model, a person only has one choice: upon meeting another person, whether or not to swap goods, i.e., trade, as a function of what good the other person is carrying. A Nash equilibrium concept is used whereby a swap will only occur voluntarily, i.e., if both people's choices are to trade.

It is relatively straightforward to characterize the set of steady-state equilibria for this model. People's decisions will depend on an expectation of what other people will do. Hence, deciding to accept a good and store it only makes sense if one's expectation is that this good will be accepted by others and that such an exchange can (eventually) lead one to be able to consume.[25] Thus, the distribution of goods holdings in the population and

[25]Here, for example, one possibility is that person 1 trades the just-produced good 1 to obtain good 3,

the associated exchange strategies of agents is key for decision making today. Existence of equilibria is made easier if randomization is allowed, but pure-strategy equilibria can occur as well. Which good(s) might appear as a medium of exchange is a function, naturally, of the relative storage costs, but also of the relative utility benefits and of consumers' rate of discount. Similarly, whether a given good is a *general* medium of exchange (and thus is accepted in trade against all other goods) depends non-trivially on the parameters of the model. Several steady-state equilibria may also exist, which is not surprising due to the central role played by expectations.

Now *fiat money* would enter this economy as another good that a subset of agents is endowed with at time zero. "Fiat" means that it is intrinsically useless: no consumer derives utility from its consumption, or has any production benefits from it. Like other goods, money is also indivisible and can be carried from period to period (in no greater quantity than one) at a utility cost $c_{\$}$. In addition, the storage properties of money are attractive: $c_{\$}$ is low. There is thus a fixed amount of money in the economy that could, potentially, function as a medium of exchange. Kiyotaki and Wright show that there are assumptions on the primitive parameters such that money, indeed, functions as a medium of exchange. Naturally, there is also another equilibrium where money has no value: it is never accepted in exchange, since it is not expected to be used in exchange in the future.

Discussion The Kiyotaki-Wright model of a medium of exchange satisfies Wallace (1998)'s dictum: the patterns of exchange emerge endogenously and non-trivially, including the role of fiat money in exchange. In the absence of public record-keeping, whereby people could get "credit" from giving up a good and hence obtain a good without exchange in the future, money at least partially plays this role: if a person comes into a meeting holding money, it must mean that that person gave up a good for money at some point in the past. The model has shortcomings, to be sure, the indivisibility of goods and money being one (relative "prices" in exchange are therefore either 1, 0, or infinity). The full absence of (centralized) markets also makes it difficult to imagine how monetary policy would be conducted. However, the original paper was followed up by many others, gradually relaxing these strong assumptions. In particular, Lagos and Wright (2005) relax all the assumptions just mentioned, yet their model remains tractable due to special assumptions made on utility functions whereby the distribution of money holdings collapses: all agents carry the same amount of money into each period.[26] To go through that framework in detail is second-year material, however. For a textbook treatment of the Lagos and Wright (2005) model and the literature that uses that framework see Rocheteau and Nosal (2017). It is also interesting to note that whereas the New Keynesian model, to be described below in Chapter 18, has been generalized in the direction of consumer and firm heterogeneity as well as a labor market with search frictions, it has not been merged with the medium-of-exchange settings. At least in part, this is because the New Keynesian model focuses on (i) cashlessness, with the argument that cash is more and more rarely used, and (ii) sticky prices, which is not the focus of the medium-of-exchange literature.

stores good 3 and then manages to trade good 3 for good 2.

[26]It is, however, difficult, using their setting, to rule out the use of bonds as a means of payment in the decentralized market. Hence the setting becomes close to a cash-in-advance model where bonds are ruled out as a means of payment by assumption.