Chapter 20

Frictional labor markets

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20.1 Introduction

Early real business cycle models assumed a frictionless labor market. In a frictionless labor market, all firms can find workers and all workers can find jobs, with the equilibrium wage equating labor demand and labor supply. The change in aggregate employment reflects shifts in either the labor demand curve or the labor supply curve.

These models ignore unemployment, that is, the phenomenon that some workers who want to work and look for jobs cannot find jobs. The unemployment rate, defined as the fraction of unemployed workers in the labor force, is an important indicator of the business cycle. The unemployment rate tends to increase when the macroeconomy is in a recession. The elevated unemployment rate is often regarded as one of the most important social costs of recessions. Various government policies, such as unemployment insurance and job training, have been implemented to reduce unemployment and address issues arising from unemployment. To analyze these policies, we need to develop a formal theoretical framework where unemployment arises endogenously.

A number of different theories are used in macroeconomics to incorporate unemployment. One simple theory is that there are frictions in wage adjustment. If the wages are too high compared to the level that clears the market, the quantity supplied can exceed the quantity demanded in the labor market. The excess supply of labor can then be interpreted as unemployment. Wages can be too high, for example, for institutional reasons such as minimum wages or unions, or there can be other reasons involving how the labor market operates. The theory of efficiency wages, for example, postulates that employers want to keep wages high so that they can induce the workers to exert a high level of effort at work.

In this chapter, we focus on unemployment arising from search frictions. In models with search frictions, it takes time, effort, and resources for workers and firms to find each other. The model incorporates the fact that it takes time for a worker to find a job that is sufficiently good for them, and it takes time for a firm to find a worker who can perform the task that the job requires. In principle, these frictions can exist in many markets, including the market for consumption goods (e.g., it may take time to find the kind of chocolate one wants to buy). Even so, we can easily imagine that this type of friction is particularly severe in the labor market because workers and jobs are heterogeneous in many dimensions. Some search models explicitly deal with the matching of heterogeneous workers and jobs. Other models treat the matching process as a "black box" and use reduced-form functions to represent it. An example of such a model is the Diamond-Mortensen-Pissarides (DMP) model, described in Section 20.4 below.

20.2 Some labor market facts

Figure 20.1 plots the unemployment rate in the postwar U.S. economy, computed from the Current Population Survey (CPS). In the statistics provided below, the entire U.S. civilian non-institutional population (16 years old and above) is divided into employment (E), unemployment (U), and not in the labor force (N). The unemployment rate is defined as the ratio of workers who cannot find a job (they are searching for a job or on temporary layoff) to the entire labor force or $U/(E+U)$. In the figure, shaded periods indicate recessions defined by the National Bureau of Economic Research.[1] The figure clearly indicates that the unemployment rate increases during recessions. There is no apparent long-run trend in the unemployment rate. Whereas the unemployment rate trended up in the 1970s, it has trended down since then, except for the stark increases during the Great Recession and the COVID-19 pandemic.

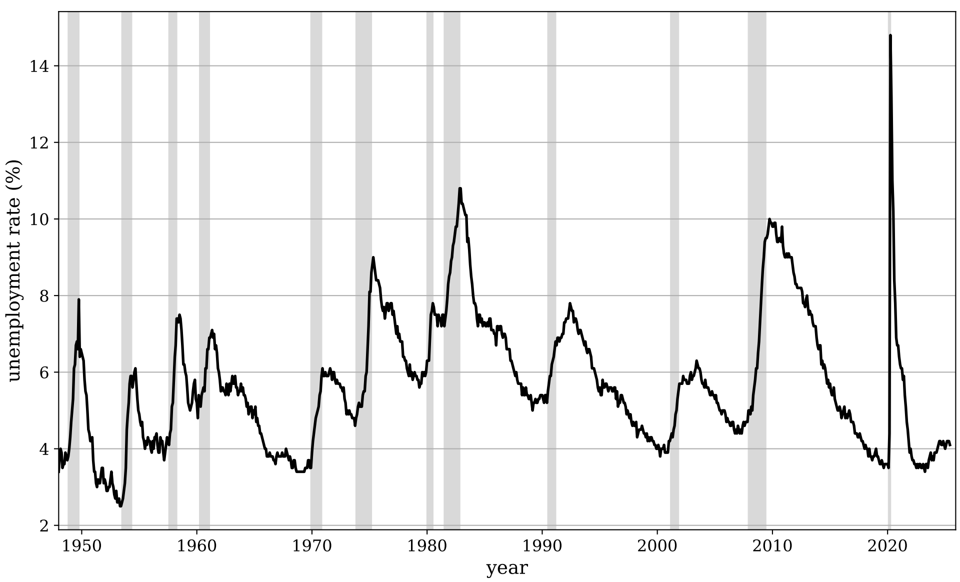

Figure 20.1: The unemployment rate in the United States.

Source: CPS.

A job opening or vacancy occurs when a firm is seeking a worker to fill an open position. Figure 20.2 plots the unemployment rate and the vacancy rate (V represents vacancy) in the United States from December 2000 to May 2022. The unemployment rate is identical to that in Figure 20.1, and the vacancy rate is computed from the Job Openings and Labor Turnover

[1] See `https://www.nber.org/research/business-cycle-dating`.

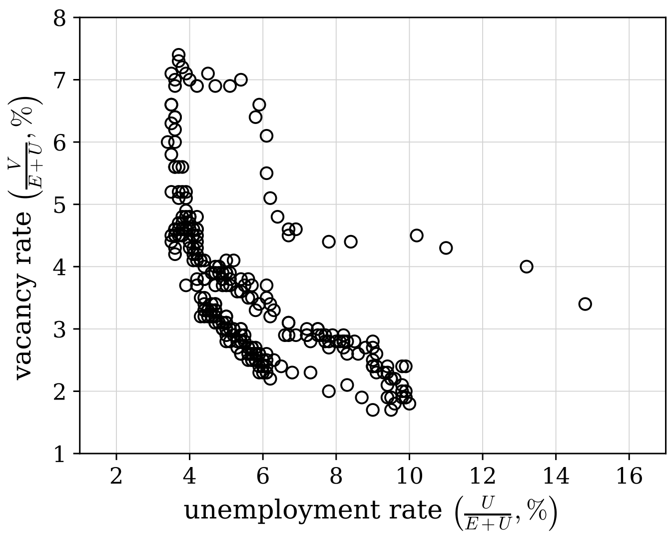

Figure 20.2: The unemployment and vacancy rates in the United States.

Source: JOLTS and CPS.

Survey (JOLTS) and the CPS.[2] Here, we simply point out that vacancies and unemployment coexist in the labor market, indicating that there are nontrivial frictions in the labor market: workers yet to find jobs and vacancies yet to find workers. We will come back to this figure later on.

20.3 A simple model of unemployment

As a starting point, consider a simple, mechanical model of unemployment. The model is schematically described in Figure 20.3. Workers are either employed (E) or unemployed (U). We normalize the total labor force to 1, and therefore

$$e_t + u_t = 1,$$

where e_t is employment in period t and u_t is unemployment in period t. We ignore the movements in and out of the labor force. Because the labor force is 1, the unemployment rate $u_t/(e_t+u_t) = u_t$, and therefore u_t is also the unemployment rate in period t. We assume unemployed workers transit into employment from one period to the next with probability λ; similarly, employed workers move into unemployment with probability σ. The probability λ is often labeled the *job finding probability* and the probability σ is called the *separation probability*.

Let the unemployment rate in period t be u_t. Then

$$u_{t+1} = (1-\lambda)u_t + \sigma(1-u_t) \tag{20.1}$$

[2]JOLTS defines the job opening rate as $V/(E+V)$. We transform it to $V/(E+U)$, which is a more relevant object for the theoretical framework below.

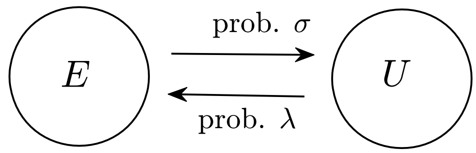

Figure 20.3: A simple model of unemployment.

holds, where the first term on the right-hand side is the unemployed workers in period t who remain unemployed in period $t+1$, and the second term is employed workers who lose their jobs between periods t and $t+1$. In steady state, where the unemployment rate is constant at $u_{t+1} = u_t = \bar{u}$,

$$\bar{u} = (1-\lambda)\bar{u} + \sigma(1-\bar{u})$$

holds, and therefore

$$\bar{u} = \frac{\sigma}{\lambda + \sigma}. \tag{20.2}$$

The steady-state unemployment rate is decreasing in the job-finding probability λ and increasing in the separation probability σ. Note that the steady state, characterized by the expression (20.2), is unique. Moreover, using (20.1) for periods t and $t-1$, we obtain

$$u_{t+1} - u_t = (1-\lambda-\sigma)(u_t - u_{t-1}),$$

and because $|1-\lambda-\sigma| \in (0,1)$, the sequence for the unemployment rate is a Cauchy sequence; thus, it converges to its steady-state value. Similarly,

$$u_{t+1} - \bar{u} = (1-\lambda-\sigma)(u_t - \bar{u})$$

holds, and we can see that $|1-\lambda-\sigma|$ determines the speed of convergence to steady state. If, for example, monthly $\lambda = 0.45$ and $\sigma = 0.034$ (the parameter values used later in the quantitative analysis), $1-\lambda-\sigma$ is almost one half, and u_t converges to the steady-state value very quickly. In this case, $\bar{u}$ can be computed to be approximately 7.0%, and if the economy starts from an unemployment rate of 15%, then after six months, the unemployment rate is already as low as 7.2%.

The job-finding rate λ is also closely linked to the average (or expected) duration of unemployment. As the probability that the unemployment duration is one period is λ, the probability that it is two periods is $(1-\lambda)\lambda$, the probability that it is three periods is $(1-\lambda)^2\lambda$, and so on, the average duration of unemployment D can be computed from

$$\begin{aligned} D &= \lambda \cdot 1 + (1-\lambda)\lambda \cdot 2 + (1-\lambda)^2\lambda \cdot 3 + \cdots \\ &= \lambda\{[1 + (1-\lambda) + (1-\lambda)^2 + \cdots] + [(1-\lambda) + (1-\lambda)^2 + \cdots] + [(1-\lambda)^2 + \cdots] + \cdots\} \\ &= \lambda\left[\frac{1}{\lambda} + (1-\lambda)\frac{1}{\lambda} + (1-\lambda)^2\frac{1}{\lambda} + \cdots\right] \\ &= \frac{1}{\lambda}. \end{aligned}$$

Thus, the average duration of unemployment is inversely related to λ. Note that D can also be derived from a recursive formulation. Let the expected duration of unemployment from the viewpoint of time t be denoted D_t. Then

$$D_t = \lambda \cdot 1 + (1-\lambda)(1 + D_{t+1}),$$

because if the worker finds a job next period (with probability λ), the duration is 1, and if she doesn't, it is one period plus the expected duration from the next period. Now note that the expected duration starting in period t is the same as that period $t+1$, and thus $D_t = D_{t+1} = D$, and we can solve the equation to yield $D = 1/\lambda$.

20.4 The Diamond-Mortensen-Pissarides (DMP) model

This section introduces a basic search and matching model often referred to as the Diamond-Mortensen-Pissarides (DMP) model. The model in this section is a discrete-time version of Pissarides (1985).[3] It is called the "search and matching model" because workers and firms have to engage in search activity (in this model, only firms engage in a costly search effort, but both the workers and the firms have to wait until they find their counterparts), and the probability of a successful search is governed by a function called the matching function.

The DMP model features costly search by firms in the form of vacancy posting. Vacancy posting is a form of investment: pay the cost now and receive payoffs later. Firms and workers share their joint surplus (the difference between being in a match and being unmatched), and the firm thus makes a profit ex post. Because firms are the only party that engages in costly search activities, this model focuses on the demand side of the labor market. The fact that workers do not have to incur costs to search, and yet share in the surplus, means that firms do not capture the full value of their investment. In this sense, this framework also clearly constitutes a departure from perfectly competitive behavior, and search equilibria may therefore not be Pareto efficient. As we will also see, the demand determination of the number of jobs is consistent with the behavior of vacancies in labor market data.

20.4.1 The matching function and labor market dynamics

Workers are either employed or unemployed. The total population is normalized to 1. The basic structure of the model is similar to that in Section 20.3, and we can view the model of the present section as endogenizing λ in the simple model there.

Firms that look for workers post vacancies to be able to search. The number of vacancies posted is endogenous—the vacancy posting behavior of firms responds to the costs and benefits of hiring workers. Vacancy posting is a (risky) investment for a firm: it is costly to post a vacancy, but if the firm successfully hires a worker, it can enjoy the profits arising from production in the future together with this worker. Unemployed workers search for firms to work for. The matching process between firms (vacancies) and unemployed workers is summarized by the *matching function*:

$$\mathcal{M}_{t+1} = M(u_t, v_t),$$

where $\mathcal{M}_{t+1}$ is the number of matches created at the beginning of period $t+1$. The function $M(\cdot,\cdot)$ is increasing in both terms and exhibits constant returns to scale. It also satisfies $M(u_t, v_t) \leq u_t$ and $M(u_t, v_t) \leq v_t$. This function is a "black box" that summarizes the complex process of firms' recruiting activities and workers' search. In particular, workers

[3] The textbook Pissarides (2000) explores various versions of the DMP model in continuous time.

and firms are heterogeneous, and it is not an easy task for a firm to find a suitable worker for its position. Different firms do not coordinate their recruiting efforts, and they may go after the same worker even when there are other workers available. The interpretation of the matching function can vary across different models, but in the basic DMP model, the black box is interpreted as incorporating all the difficulties involved in matching.

We assume that search is random, that is, all vacancies have the same chance of finding workers, and all workers have the same chance of finding a vacancy. Thus the probability that a worker meets a firm is

$$\frac{M(u_t, v_t)}{u_t} = M\left(1, \frac{v_t}{u_t}\right) = M(1, \theta_t),$$

where θ_t is defined as $\theta_t \equiv v_t/u_t$ and often referred to as the labor market tightness. Let us use the notation

$$\lambda_w(\theta_t) \equiv M(1, \theta_t). \tag{20.3}$$

This $\lambda_w(\theta_t)$ corresponds to λ in Section 20.3. Note that $\lambda_w(\cdot)$ is increasing in θ_t from our assumptions about the matching function. The probability that a vacant firm meets a worker is

$$\frac{M(u_t, v_t)}{v_t} = M\left(\frac{u_t}{v_t}, 1\right) = M\left(\frac{1}{\theta_t}, 1\right).$$

Let us define

$$\lambda_f(\theta_t) \equiv M\left(\frac{1}{\theta_t}, 1\right).$$

Note also that

$$\lambda_w(\theta_t) = \theta_t \lambda_f(\theta_t).$$

It turns out that, under the assumption we will make later (the production in a match is sufficiently large), all firms and workers accept all matches once they meet. Thus $\lambda_w(\theta_t)$ represents the job-finding probability of an unemployed worker. It also represents the probability that a worker transitions from unemployment to employment, and unlike in Section 20.3, it depends on the labor market tightness. In this section (as in Section 20.3), we assume that matches dissolve with probability $\sigma \in (0, 1)$. Therefore, the dynamics of unemployment are governed by

$$u_{t+1} = (1 - \lambda_w(\theta_t))u_t + \sigma(1 - u_t). \tag{20.4}$$

The first term on the right-hand side is the unemployed workers at period t who stay unemployed at period $t + 1$. The second term is the employed workers (note that $e_t = 1 - u_t$) who separate from their jobs between t and $t + 1$.

When the number of vacancies is constant at v (which will be the steady-state of the model when productivity is constant over time), it is straightforward to show that there exists a unique steady-state value of u_t (call it $\bar{u}$). To see this fact, set $u_{t+1} = u_t = \bar{u}$ in (20.4) and obtain

$$\bar{u} = \frac{\sigma}{\lambda_w(v/\bar{u}) + \sigma}. \tag{20.5}$$

This equation can be rewritten as, using (20.3),

$$M(v, \bar{u}) + \sigma\bar{u} = \sigma. \tag{20.6}$$

Because the right-hand side is constant and the left-hand side is increasing in $\bar{u}$, the solution for $\bar{u}$ in (20.5) is unique. Further note that (20.6) describes a negative relationship between v and $\bar{u}$ when σ is kept constant.

Now let us go back to Figure 20.2. Figure 20.4 connects the data points of Figure 20.2. It is clear that there is a negative relationship between the unemployment rate and the vacancy rate. This regularity is often called the *Beveridge curve*. The Beveridge curve relationship is consistent with the equation (20.6), and therefore provides support for this component of the DMP model. For this reason, equation (20.6) is often referred to as the Beveridge curve relationship. Moreover, the strong procyclical movement of vacancies indicates that the firm's recruiting activities (i.e., the labor demand movements) play an important role in driving the cyclical movement of the unemployment rate.

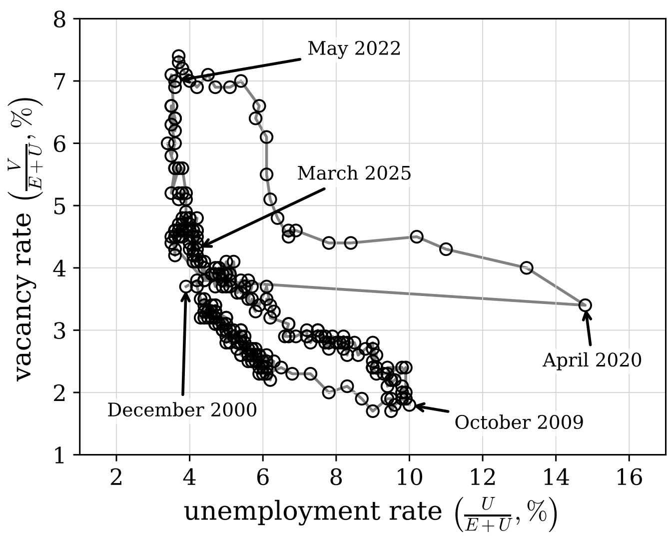

Figure 20.4: Beveridge curve in the United States.

Sources: JOLTS and CPS.

20.4.2 Market equilibrium under endogenous vacancy creation

Let us now look at how the number of vacancies, v_t, is determined. Here, we assume the production is conducted by a match between one firm (vacancy) and one worker. For the moment, we assume that firms do not use capital and assume that the match between a firm and a worker can produce z_t units of goods. We further assume that, as in the standard real business cycle (RBC) model, there are aggregate movements in productivity. In particular, we assume that z_t varies stochastically over time according to a first-order Markov process.

It turns out that we can formulate the model recursively and the relevant state variable in the general equilibrium is only z. Assume that the firm discounts the future profit with the discount factor $\beta \in (0, 1)$. Because future profits enter linearly in the firm's objective function, we are also assuming that firms are risk-neutral. The Bellman equation for a firm that has already matched with a worker is

$$J(z) = z - w(z) + \beta \mathbb{E}[(1 - \sigma) J(z') + \sigma V(z')], \tag{20.7}$$

where $J(z)$ is the value of a matched firm. The flow value $z - w(z)$ constitutes profits, where $w(z)$ is the wage paid to the worker when the aggregate productivity is z. The parameter $\beta \in (0, 1)$ is the discount factor of the firm (which will be identical to the worker's discount factor), and $\mathbb{E}[\cdot]$ indicates the expected value (conditional on the current period information). The prime ($'$) indicates the next period. $V(z)$ represents the value of a vacancy.

The Bellman equation for a vacant firm is

$$V(z) = -\kappa + \beta\mathbb{E}[\lambda_f(\theta)J(z') + (1 - \lambda_f(\theta))V(z')], \tag{20.8}$$

where $\kappa > 0$ is the cost of posting a vacancy. We also assume that anyone can set up a vacancy and enter the market ("free entry"). Thus, in equilibrium with a strictly positive level of vacancies (which we will assume throughout), the value of a vacancy is driven down to zero:

$$V(z) = 0. \tag{20.9}$$

(20.8) and (20.9) imply

$$\frac{\kappa}{\lambda_f(\theta)} = \beta\mathbb{E}[J(z')]. \tag{20.10}$$

Intuitively, the cost of creating a vacancy has to be equal to the expected value of the future filled job, $\beta\mathbb{E}[J(z')]$, times the probability of finding a worker, $\lambda_f(\theta)$.

To determine the equilibrium wage, we first have to consider the worker side. We assume that a worker is infinitely-lived, consumes what she receives every period, and has a linear utility function with discount factor β (i.e., the same discount factor as used by the firm):[4]

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t c_t\right].$$

The Bellman equation for an employed worker is

$$W(z) = w(z) + \beta\mathbb{E}[(1 - \sigma)W(z') + \sigma U(z')], \tag{20.11}$$

where $W(z)$ is the value of an employed worker and $U(z)$ is the value of an unemployed worker. We assume that an unemployed worker receives a constant amount of goods $b < z_t$ (which has to hold for all possible values of z_t). b can be interpreted as home production or unemployment insurance. The Bellman equation for an unemployed worker is

$$U(z) = b + \beta\mathbb{E}[\lambda_w(\theta)W(z') + (1 - \lambda_w(\theta))U(z')]. \tag{20.12}$$

Once a firm and a worker match, they are in a *bilateral monopoly* situation: the only possible seller (of the labor service) for the firm is the worker it matched with, and the only

[4] Because the firms in this economy can earn a profit (in particular, the aggregate profit is positive in the steady-state of the economy), there is a question of who receives the profit (i.e., the ownership of the firms). Here, we implicitly assume that the firms are owned by someone outside the economy who has the same discount factor as the worker. Alternatively, we can assume that the firm is owned by workers. As we will see later in this section, this result follows because only the *difference* in income between the employed worker and the unemployed worker matters for the equilibrium dynamics of unemployment. With linear utility, there are no reasons for workers to trade the firm ownership (stocks). In Section 20.8, we assume a closed economy and make this stock holding explicit.

possible buyer for the worker is the firm she matched with. In such a situation, we cannot use the marginal principle to determine the wage because there is no competition. The match generates a surplus. In the current period, the match jointly generates z; if they separate, the worker obtains b, and the firm nothing. Therefore, it is jointly beneficial for the firm and the worker to be together because of the assumption $z > b$. Unless they are separated, this flow surplus $z - b$ is generated in the future as well. We assume that the firm and the worker split the surplus following the *generalized Nash bargaining* rule. This Nash bargaining rule splits the surplus so that the Nash product, which is the product of the surpluses of each party (in our case, those of the firm and the worker), is maximized. The generalized Nash bargaining rule uses the weighted product instead, where the "weight" is represented as the exponent to each of the surpluses.

In our formulation, the generalized Nash bargaining solution solves

$$\max_{w} \ (\tilde{W}(w, z) - U(z))^{\gamma} (\tilde{J}(w, z) - V(z))^{1-\gamma}, \tag{20.13}$$

where $\tilde{W}(w, z)$ is the value of an employed worker when the current wage is w:

$$\tilde{W}(w, z) \equiv w + \beta \mathbb{E}[(1 - \sigma) W(z') + \sigma U(z')]. \tag{20.14}$$

Note that the Bellman equation (20.11) assumes that the wage is the equilibrium value under z, $w(z)$. Here, in (20.14), we are allowing w to be any value. Therefore $\tilde{W}(w(z), z) = W(z)$ holds. Similarly, $\tilde{J}(w, z)$ is the value of a job matched with a worker when the wage is w:

$$\tilde{J}(w, z) \equiv z - w + \beta \mathbb{E}[(1 - \sigma) J(z') + \sigma V(z')].$$

In (20.13), the worker's surplus is $\tilde{W}(w, z) - U(z)$, and the firm's surplus is $\tilde{J}(w, z) - V(z)$. The exponent $\gamma \in (0, 1)$ represents the "weight" on the worker's surplus. It is often referred to as the "bargaining power" of the worker. By taking the first-order condition and rearranging, we find the equation that w has to satisfy:

$$(1 - \gamma)(\tilde{W}(w, z) - U(z)) = \gamma(\tilde{J}(w, z) - V(z)). \tag{20.15}$$

The detailed derivation of (20.15) is in Appendix 20.A.1.

The six equations (20.7), (20.8), (20.9), (20.11), (20.12), and (20.15) define the equilibrium. These can be rearranged to obtain a difference equation in θ_t.[5] The result is

$$\frac{\kappa}{(1 - \gamma)\lambda_f(\theta_t)} = \beta \mathbb{E}\left[z_{t+1} - b + \frac{1 - \sigma - \gamma \lambda_w(\theta_{t+1})}{1 - \gamma} \frac{\kappa}{\lambda_f(\theta_{t+1})} \right]. \tag{20.16}$$

When z is constant over time, the steady-state value of θ_t (denote it $\bar{\theta}$) solves

$$\frac{\kappa}{(1 - \gamma)\lambda_f(\bar{\theta})} = \beta \left[z - b + \frac{1 - \sigma - \gamma \lambda_w(\bar{\theta})}{1 - \gamma} \frac{\kappa}{\lambda_f(\bar{\theta})} \right]. \tag{20.17}$$

This equation determines $\bar{\theta} = \bar{v}/\bar{u}$, where $\bar{v}$ is the steady-state value of vacancy. Multiplying by $\lambda_f(\bar{\theta})$ on both sides, we see that the right-hand side is decreasing in $\bar{\theta}$, ensuring the uniqueness of a $\bar{\theta}$ that satisfies this equation. Often this condition is called the job creation

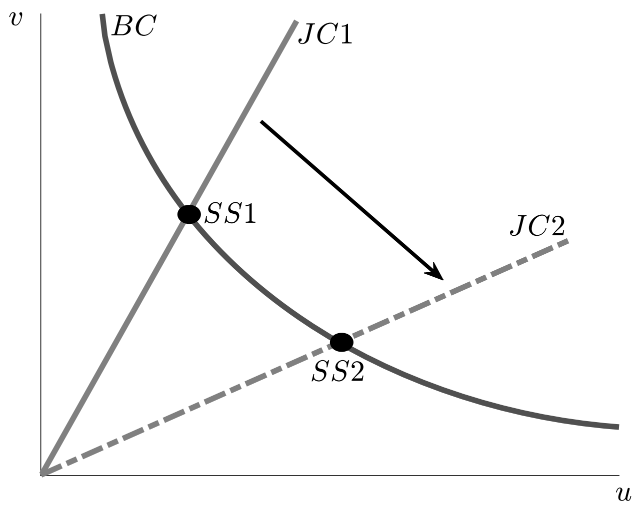

Figure 20.5: The determination of the steady state

condition. The job creation condition, together with the Beveridge curve relationship (20.6), determine $\bar{v}$ and $\bar{u}$.

Figure 20.5 describes the steady-state determination of v and u. The BC curve represents (20.6), which describes the steady-state relationship between u and v. The straight lines $JC1$ and $JC2$ represent the relationship between u and v that correspond to different values of θ. When $JC1$ represents the value of θ that satisfy (20.17), the steady-state values of u and v are at $SS1$. When z goes down, from (20.17) we can see $\bar{\theta}$ also goes down. The JC line shifts from $JC1$ to $JC2$. In the new steady-state ($SS2$), v is smaller and u is larger.

The transition dynamics are also easy to analyze. Consider an unanticipated one-time permanent decline in z. Because the decline is permanent, the new job creation equation holds with a new steady-state value of θ. In other words, the equation (20.17) holds with the new value of $\bar{\theta}$. In Figure 20.6, the new value of $\bar{\theta}$ is represented by the new line $JC2$. Under a mild condition,[6] it can be shown that the jump to the new value of $\bar{\theta}$ has to be immediate. This result follows because θ_t would diverge away from the new steady state unless θ_t immediately jumps to the new steady-state value.[7] Because u cannot jump, v immediately drops so that v/u becomes the new value of $\bar{\theta}$. After that, the economy gradually converges to the new steady state ($SS2$) along the $JC2$ line.

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[5] We can analyze the implications on equilibrium wages using the same set of equations. The analysis of the wages is in Appendix 20.A.2.

[6] For example, $(1-\sigma)\eta(u_t, v_t) - \gamma\lambda_w(\theta_t) > 0$ for all t, where $\eta(u_t, v_t)$ will be defined below in (20.21), is sufficient. This condition is always satisfied when the time period is short, that is, σ and $\lambda_w(\theta_t)$ are both close to zero.

[7] First, for a constant z, there is always—as argued above—a unique constant solution for $\bar{\theta}$ to the difference equation. Second, using equation (20.16), it is possible to show that the dynamics are unstable and therefore θ_t diverges if it is not equal to $\bar{\theta}$. This fact can be shown in two steps. First, the above condition in footnote 6 ensures that the right-hand side of (20.16) is increasing in θ_{t+1} and therefore θ_{t+1} is increasing in θ_t. Second, it is always the case (again, under the above condition) $d\theta_{t+1}/d\theta_t > 1$ around the unique steady state.

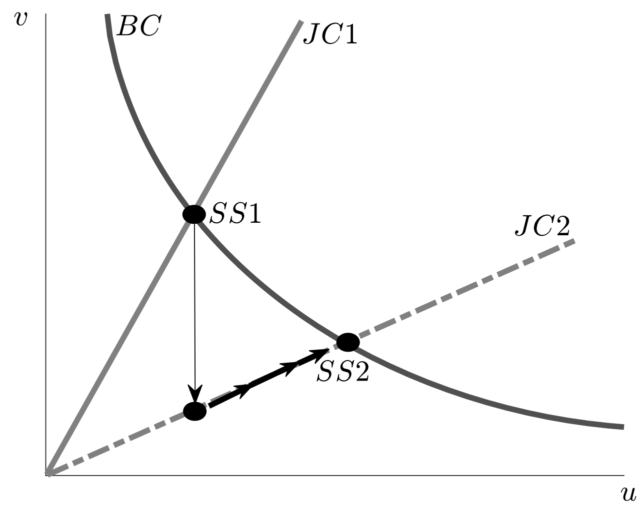

Figure 20.6: The transition dynamics of the DMP model.

20.4.3 Efficiency

Unemployment has been considered one of the most important macroeconomic challenges and various policies have been proposed and implemented to reduce unemployment. The DMP model describes the evolution of the labor market through the lens of the firm entry decision and costly vacancy posting. It is possible to lower unemployment by posting more vacancies, but the costs of doing so may exceed the benefits. Before considering policies, it is critical to know what kind of inefficiencies are present in the economy and whether unemployment in the market equilibrium is too high or too low compared to the social optimum. The question of efficiency can be phrased as asking if the private returns to firm entry match the social return from entry. Here, in the main text, we will give an intuitive discussion of the efficiency considerations and relegate a formal proof to Appendix 20.A.3.[8]

Let us begin by reducing the decentralized equilibrium to two equations. Define the *expected surplus of a match* as

$$S_t \equiv \mathbb{E}_t[W(z_{t+1}) - U(z_{t+1}) + J(z_{t+1}) - V(z_{t+1})],$$

where the expectation is taken at time t. From (20.9), (20.10), and (20.15) (which implies $(1-\gamma)S_t = \mathbb{E}_t[J(z_{t+1})]$), we obtain

$$\kappa = \lambda_f(\theta_t)(1-\gamma)\beta S_t. \tag{20.18}$$

This equation (20.18) characterizes the private incentives for vacancy posting by firms: the cost of a vacancy is κ and the benefit of vacancy is the probability of finding a worker times the firm's share of the surplus generated by a match. Note that a higher γ gives more of the surplus to the worker and reduces the firm's incentive to post vacancies.

[8] Also see Fukui and Mukoyama (2025) for a formal treatment and the case with on-the-job search.

From (20.7), (20.11), and (20.12) all for time $t+1$ (and taking expectations at time t), and using (20.9) and (20.15) (which implies $\mathbb{E}_{t+1}[W(z_{t+2}) - U(z_{t+2})] = \gamma S_{t+1}$),

$$S_t = \mathbb{E}_t \left[z_{t+1} - b + \beta S_{t+1}(1 - \sigma - \gamma \lambda_w(\theta_{t+1})) \right]. \tag{20.19}$$

This equation shows that the private value of a match reflects the expected increase in production beyond home production plus a continuation value. Note that the continuation value is lower if the match is likely to dissolve (high σ). Note also that the continuation value of this match is smaller if an unemployed worker has a higher chance of finding a job in the next period.

Now consider the social planner's incentives to create vacancies. When the planner chooses the number of vacancies at time t, there are implications at different points in time. At time t there is the vacancy cost κ. On the other side, the most immediate benefit is that there will (probabilistically) be more matches at $t+1$, each delivering an increase in production by $z_{t+1} - b$. Suppose, for a moment, the economy ended after date $t+1$. This assumption would imply a first-order condition for the planner which reads

$$\kappa = M_2(u_t, v_t)\beta(\mathbb{E}_t z_{t+1} - b),$$

where $M_2(u, v)$ represents the partial derivative of the matching function with respect to the second argument. The right-hand side of this equation shows the marginal increase in matches and the expected benefit of each match. For the decentralized equilibrium, the joint surplus S_t would be simply $\mathbb{E}_t z_{t+1} - b$ so (20.18) becomes

$$\kappa = \lambda_f(\theta_t)(1 - \gamma)\beta(\mathbb{E}_t z_{t+1} - b).$$

So, in this example, the decentralized equilibrium will coincide with the planner's choice if $M_2(u_t, v_t) = \lambda_f(\theta_t)(1 - \gamma)$ or, using the definition of $\lambda_f(\theta_t)$

$$M_2(u_t, v_t) = \frac{M(u_t, v_t)}{v_t}(1 - \gamma). \tag{20.20}$$

The ratio $M(u_t, v_t)/v_t$ is the "average product" of a vacancy, which is higher than the marginal product of a vacancy. The planner recognizes that adding a vacancy reduces the probability that the existing vacancies will result in a match. The fact that the private firms ignore this effect is a form of externality called a congestion externality. In this sense, the market overvalues the creation of a vacancy by neglecting this negative spillover. On the other hand, the firm only captures part of the surplus from a match—a share $1 - \gamma$. The decentralized equilibrium will be efficient if these two considerations offset, which is the case when condition (20.20) holds. Define $\eta(u_t, v_t)$, often called the *elasticity of the matching function*, such that

$$1 - \eta(u_t, v_t) \equiv \frac{M_2(u_t, v_t)v_t}{M(u_t, v_t)}. \tag{20.21}$$

Then the condition above becomes $\eta(u_t, v_t) = \gamma$ or in words: the elasticity of the matching function with respect to vacancies is equal to the firm's bargaining power $(1 - \gamma)$.

In our dynamic model, the vacancy-creation decision (which can be viewed as an investment decision) at date t has payoffs that go beyond $\mathbb{E}_t z_{t+1} - b$ at time $t+1$. First, there is the chance that the match persists yielding further gains in production at $t+2$, $t+3$, and so on. Second, a worker in a match is not available to meet other firms and form other matches. The foregone opportunities to search is an opportunity cost of a match. The opportunity cost at $t+1$ is valued by the planner as the discounted expected value of $M_1(u_{t+1}, v_{t+1})$ times the value of a match formed at $t+1$, where $M_1(u, v)$ represents the partial derivative of the matching function with respect to the first argument. On the other hand, the market values the opportunity cost as $\mathbb{E}_t \lambda_w(\theta_{t+1}) \beta \gamma S_{t+1}$ as shown in (20.19). The market only values a fraction γ of the total surplus as the opportunity cost. We again have a congestion externality in that the worker privately values the opportunity cost using the average product $\lambda_w(\theta_{t+1})$ while the planner uses $M_1(u_{t+1}, v_{t+1})$. Using similar steps as in the previous paragraph we obtain a condition for efficiency of

$$\frac{M_1(u_{t+1}, v_{t+1}) u_t}{M(u_{t+1}, v_{t+1})} = \gamma. \tag{20.22}$$

The worker undervalues the opportunity cost by only considering a share γ of the surplus but neglects the fact that removing one worker from the pool of searching workers makes it more likely that other workers will find a match. When (20.22) holds, these two considerations offset and the market appropriately values the opportunity cost of foregone search. The same condition also applies at future dates while the worker remains employed to account for the opportunity cost of foregone search at those dates.

Suppose we have a constant returns to scale Cobb-Douglas matching function of the form

$$M(u_t, v_t) = u_t^{\eta} v_t^{1-\eta},$$

where η is now a parameter. By setting $\eta = \gamma$, we can satisfy both the conditions (20.20) and (20.22) and the decentralized equilibrium is efficient. We can actually generalize this condition to any constant returns to scale matching function and the restriction $\eta(u_t, v_t) = \gamma$. This result was shown by Hosios (1990) and is often called the *Hosios condition*. For a full proof of this result, see Appendix 20.A.3.

20.5 Labor market facts, once again

The modern macroeconomic study of the labor market also considers the gross flows behind the movement of stocks (e.g., unemployment). In fact, the model we presented in Sections 20.3 and 20.4 provides an analysis of gross flows between the state E (employment) and the state U (unemployment).

More generally, empirical studies typically focus on three states of the labor market, E (employment), U (unemployment), and N (not in the labor force). Thus we can consider six flows across these states, as described in Figure 20.7.

Figures 20.8, 20.9, and 20.10 plot these flow rates, from Fallick and Fleischman (2004).[9] For example, the E to U flow rate in Figure 20.8 plots the fraction of employed workers that flow into U in the following month. One can see how the gross flows influence the movement

[9] The data is from `https://www.federalreserve.gov/pubs/feds/2004/200434/200434abs.html`. Seasonal adjustment is made using X-13 ARIMA-SEATS from the U.S. Census Bureau.

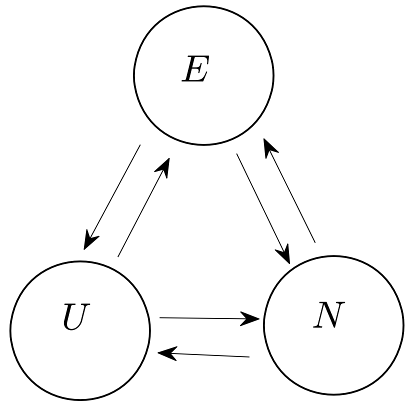

Figure 20.7: Flows among three states.

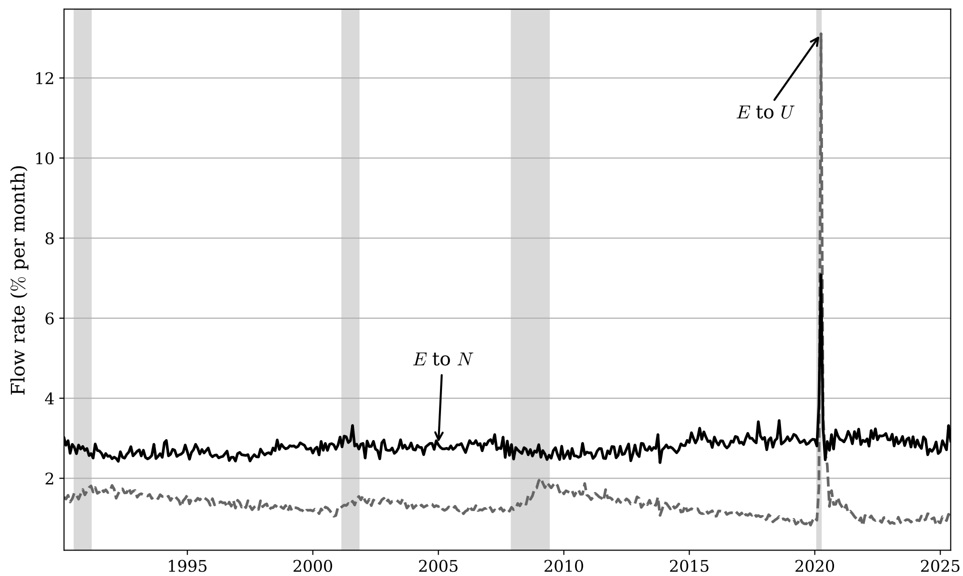

Figure 20.8: Flow rates out of E.

Sources: CPS (Fallick and Fleischman, 2004).

of stocks. For example, U increases in recessions because the inflows from E and N into U go up and the outflows to E and N go down.

These stylized facts provide important information for building the models of unemployment. Here, we point out two simple observations. First, the flow rate from U to E is strongly procyclical. This property is consistent with the DMP model in Section 20.4. Second, the flow rate from E to U is strongly countercyclical. This fact implies the constant separation rate in the DMP model above is not consistent with the data. We will come back to this point in Section 20.7.

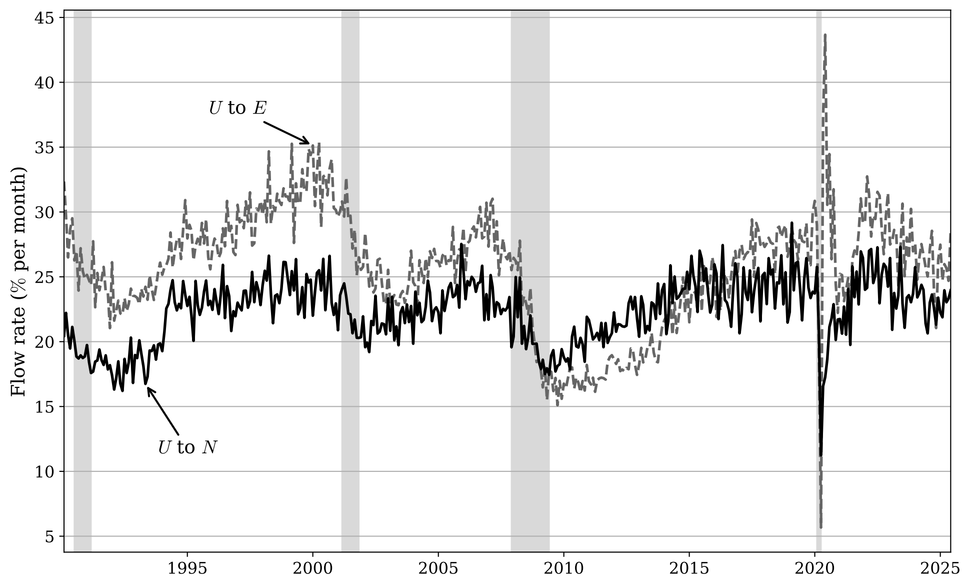

Figure 20.9: Flow rates out of U.

Sources: CPS (Fallick and Fleischman, 2004).

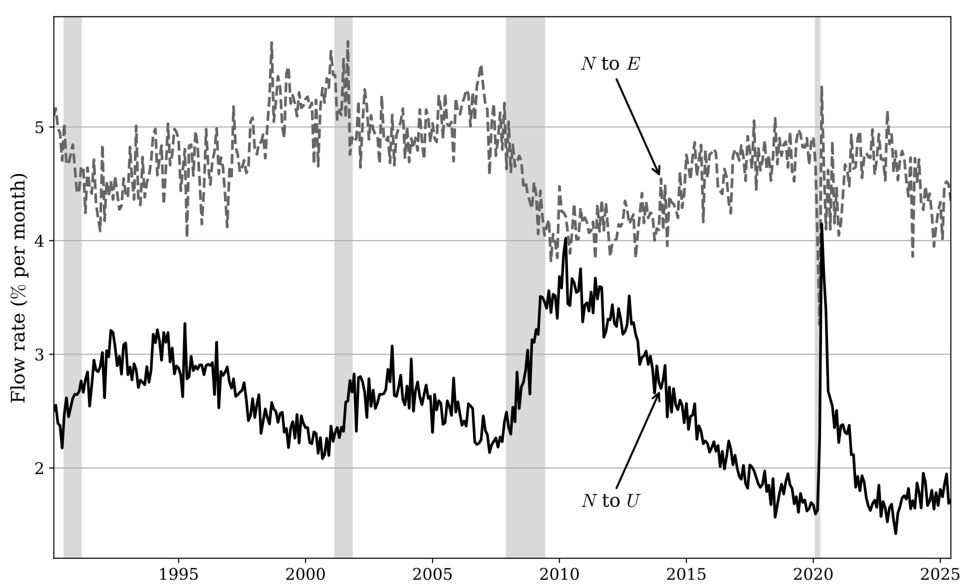

Figure 20.10: Flow rates out of N.

Sources: CPS (Fallick and Fleischman, 2004).

20.6 The unemployment volatility puzzle

We now examine the quantitative performance of the DMP model that we presented in Section 20.4. Let us go back to the difference equation (20.16). We will thus evaluate the model (as in the analysis of the RBC models) by assigning functional forms and parameter values to the model.

Assume that the matching function is specified as the Cobb-Douglas form,

$$M(u, v) = \chi u^{\eta} v^{1-\eta}, \tag{20.23}$$

where $\eta \in (0, 1)$. Then $\lambda_w(\theta) = \chi\theta^{1-\eta}$ and $\lambda_f(\theta) = \chi\theta^{-\eta}$. Note that we are using the same notation η as in Section 20.4.3. If we compute the (negative of) the elasticity of $\lambda_f(\theta)$ with respect to θ, it is exactly the constant value η in this Cobb-Douglas form.

In addition, specify the process of z_t as

$$\hat{z}_{t+1} = \rho\hat{z}_t + \varepsilon_{t+1}, \tag{20.24}$$

where the hat $(\hat{})$ describes the log-deviation: $\hat{z}_t = \log(z_t) - \log(\bar{z})$ ($\bar{z}$ is the steady-state value of z). The parameter $\rho \in (0, 1)$ represents the persistence of the shock, and the i.i.d. shock $\varepsilon_{t+1} \sim N(0, \sigma_{\varepsilon}^2)$. Here, $\log(\bar{z})$ is normalized to zero.

20.6.1 The log-linearized solution

We will solve this model by log-linearizing the solution around the steady state. Log-linearizing equation (20.16) yields

$$\mathcal{A}\hat{\theta}_t = \mathbb{E}[\bar{z}\hat{z}_{t+1} + \mathcal{B}\hat{\theta}_{t+1}], \tag{20.25}$$

where

$$\mathcal{A} = \frac{\kappa\bar{\theta}^{\eta}\eta}{(1-\gamma)\beta\chi}$$

and

$$\mathcal{B} = \frac{1-\sigma}{1-\gamma}\frac{\kappa\bar{\theta}^{\eta}\eta}{\chi} + \frac{\gamma\kappa\bar{\theta}}{1-\gamma}.$$

Appendix 20.A.4 briefly describes the basic method of log-linearization. Further details can be found, for example, in Uhlig (2001).

To solve the difference equation (20.25) using the method of undetermined coefficients, first guess $\hat{\theta}_t = \mathcal{C}\hat{z}_t$, where $\mathcal{C}$ is the undetermined coefficient. Inserting this guess into (20.25) and using $\mathbb{E}[z_{t+1}] = \rho z_t$, we obtain

$$\mathcal{C} = \frac{\rho}{\mathcal{A} - \rho\mathcal{B}}.$$

Rearranging, we obtain the relationship between $\hat{\theta}_t$ and $\hat{z}_t$ as

$$\hat{\theta}_t = (1-\gamma)\left[\frac{\kappa\bar{\theta}^{\eta}\eta}{\chi}\left(\frac{1}{\rho\beta} - (1-\sigma)\right) + \kappa\gamma\bar{\theta}\right]^{-1}\hat{z}_t. \tag{20.26}$$

Analytically, this result provides important insights into the model's performance relative to the data. In particular, (20.26) implies (for a given $\bar{\theta}$), a smaller κ, a smaller γ, a smaller η, a larger χ, a larger ρ, a larger β, or a smaller σ makes the response of θ_t larger.

Table 20.1: Parameter values

Calibrated Parameters	Value
β	0.996
ρ	0.949
σ_ε	0.0065
σ	0.034
χ	0.45
b	0.4
γ	0.72
η	0.72

20.6.2 Calibration

Now we set the relevant parameter values. Let one period in the model be one month, so that we can use the U.S. labor market data and capture the fast labor market dynamics in the U.S. economy. Here, we assume the parameter values in Table 20.1.

The discount factor β is set at $0.947^{\frac{1}{12}} = 0.996$. The annual value of 0.947 is taken from the standard real business cycle literature (Cooley and Prescott, 1995).

The parameters for the process for $\hat{z}$, ρ, and σ_ε, are taken from Hagedorn and Manovskii (2008) and adjusted to a monthly frequency. Here, σ_ε is the standard deviation of ε_{t+1} in (20.24), assuming that ε_{t+1} follows a normal distribution. Hagedorn and Manovskii (2008) estimate this process from seasonally adjusted (quarterly) real average output per person in the nonfarm business sector in the US.

The values $b = 0.4$, $\eta = 0.72$, and $\gamma = 0.72$ follow Shimer (2005). He justifies the value of b from the replacement rate of the unemployment benefit in the US. The value of η comes from estimating a matching function using the US data. He sets $\eta = \gamma$ so that the Hosios condition is satisfied. Furthermore, following Shimer (2005), we set the parameter value for κ so that the equation (20.17) is satisfied in steady state, with the normalization $\bar{\theta} = 1$. The values of χ and σ also follow Shimer (2005).[10] He computes these values from the US labor market data, using the outflow rate from unemployment and inflow rate into unemployment.

20.6.3 Quantitative results

With the given parameter values and the equations (20.4) and (20.26), we can simulate the model by randomly generating the series of $\hat{z}_t$ following (20.24). Table 20.2 is the summary of the U.S. data that we will compare the model against. Here, z measures labor productivity. All are originally from monthly data but averaged to quarterly data, and logged and HP-filtered with the smoothing parameter of 1600. The table is taken from Hagedorn and Manovskii (2008).

[10] These values are larger than the UE flow rates and the EU flow rates in Figures 20.8 and 20.9. This discrepancy is because Shimer (2005) calculates the outflow from and the inflow into U (which include the flows in and out of N), instead of UE and EU flow rates. His methodology does not require matching workers from one month to the next; instead, it utilizes the duration distribution of unemployment.

Table 20.2: Summary statistics for quarterly U.S. data

		u	v	v/u	z
Standard Deviation		0.125	0.139	0.259	0.013
Quarterly Autocorrelation		0.870	0.904	0.896	0.765
	u	1	−0.919	−0.977	−0.732
Correlation Matrix	v	—	1	0.982	0.460
	v/u	—	—	1	0.967
	z	—	—	—	1

Table 20.3: Model statistics

		u	v	v/u	z
Standard Deviation		0.005	0.016	0.020	0.013
Quarterly Autocorrelation		0.826	0.700	0.764	0.765
	u	1	−0.839	−0.904	−0.804
Correlation Matrix	v	—	1	0.991	0.972
	v/u	—	—	1	0.961
	z	—	—	—	1

Table 20.3 contains the statistics from the model-generated data. The model generates the right correlations between variables, but the magnitude of the fluctuations in u, v, and θ is too small compared to the data. This discrepancy is often referred to as the *unemployment volatility puzzle* or the *labor market volatility puzzle* (or the "Shimer puzzle," after Shimer (2005)). The model's inability to match the magnitude of labor market fluctuations triggered an extensive body of research in early 2000s.

Intuitively, there are two reasons, corresponding to benefits and costs of hiring, for the quantitatively small response. First, the benefit of hiring a worker is procyclical, but the magnitude is not large. One reason is that the wage increases in booms, and it weakens the response of profit to the productivity shock. Second, the cost of hiring a worker, $\kappa/\lambda_f(\theta)$ moves together with θ. In booms, θ increases, and this increase in cost dampens the firm's response to a positive productivity shock.

20.6.4 Rigid wages

Many possible "solutions" are proposed for the unemployment volatility puzzle. Although there is no clear consensus among researchers in terms of which proposed "solution" is the most plausible one, here we highlight the role of rigid wages. As we explained above, the response of wages to productivity shocks dampens the volatility of profits. Rigid wages would make the benefit of creating a vacancy more volatile.

Empirically, there have been many studies about wage rigidity, both nominal and real.[11] Even without search frictions, rigid wages can generate unemployment by preventing the

[11] See, for example, McLaughlin (1994), Elsby and Solon (2019), Jardim, Solon, and Vigdor (2019), Kurmann and McEntarfer (2019), and Grigsby et al. (2021).

labor market from clearing. Here, wage rigidity changes the incentive for the firms to hire workers in booms and recessions.[12]

Instead of the generalized Nash bargaining, here we assume that wages are rigid at the steady-state value $w = \bar{w}$. Combining (20.7) with (20.10), we obtain

$$\frac{\kappa}{\lambda_f(\theta_t)} = \beta\mathbb{E}\left[z_{t+1} - \bar{w} + \frac{(1-\sigma)\kappa}{\lambda_f(\theta_{t+1})}\right].$$

Log-linearizing,

$$\hat{\theta}_t = \left[\frac{\kappa\bar{\theta}^{\eta}\eta}{\chi}\left(\frac{1}{\rho\beta} - (1-\sigma)\right)\right]^{-1}\hat{z}_t. \tag{20.27}$$

With the same calibration, the model outcome can be computed as in Table 20.4. Unemployment fluctuations are of the same magnitude as the data. Comparing (20.26) and (20.27) reveals two factors that the wage rigidity can make the firm's profit more volatile. First, the latter is not multiplied by $(1-\gamma)$, indicating that the additional gain in the surplus is no longer shared between the worker and the firm, and the firm can receive all the additional gain. Second, the term $\kappa\gamma\bar{\theta}$, which represents the improvement of the worker's bargaining position due to the rise in the future job finding probability, is absent when wages are rigid.

Therefore, one can see that a (real) wage rigidity can address the volatility puzzle. We note, however, the sources and magnitude of the wage rigidity still remains an active area of research.

Table 20.4: Model statistics with fixed wages

		u	v	v/u	z
Standard Deviation		0.115	0.329	0.425	0.013
Quarterly Autocorrelation		0.825	0.693	0.763	0.765
Correlation Matrix	u	1	−0.791	−0.881	−0.784
	v	—	1	0.986	0.969
	v/u	—	—	1	0.961
	z	—	—	—	1

20.7 Endogenous separations

The previous sections focused on the role of fluctuations in the job-finding probability. As we saw in Section 20.5, both job-finding and separation rates are strongly cyclical.[13] In particular, at the onset of recessions, the increase in the separation rate tends to cause a sharp increase in the unemployment rate. In Figure 20.8, the *EU* flow rate increases in recessions, and the magnitude of the increase depends on the severity of the recession, suggesting that the separation rate changes endogenously with the business cycle. In this section, we extend the basic model in Section 20.4 to allow for endogenous separation.

[12] The role of wage rigidity in this context was first explored by Hall (2005) and Shimer (2005).

[13] Fujita and Ramey (2009) emphasize the importance of the separation margin in unemployment fluctuations.

20.7.1 Model extension

We now assume the firm has to pay a cost for maintaining the match, $c(\sigma)$ and σ is a choice variable for the firm. The cost increases if the firm wants to make the separation probability small, that is, $c'(\sigma) < 0$.

The matched firm's Bellman equation is now

$$J(z) = \max_{\sigma} \; z - w(z) - c(\sigma) + \beta \mathbb{E}\left[(1-\sigma)J(z') + \sigma V(z')\right].$$

The rest of the equilibrium conditions ((20.8), (20.9), (20.11), (20.12), and (20.15)) are the same as in the Section 20.4. The optimal value of σ is now a function of z (denote it as $\sigma(z)$). The unemployment dynamics are, therefore,

$$u_{t+1} = (1 - \lambda_w(\theta_t(z_t)))u_t + \sigma(z_t)(1 - u_t).$$

The first-order condition for σ is, using (20.9),

$$-c'(\sigma) = \beta \mathbb{E}[J(z')].$$

From (20.10), this equation implies

$$-c'(\sigma) = \frac{\kappa}{\lambda_f(\theta)}. \tag{20.28}$$

The job creation condition can be derived in the same manner as in the exogenous separation case:

$$\frac{\kappa}{(1-\gamma)\lambda_f(\theta_t)} = \beta \mathbb{E}\left[z_{t+1} - b - c(\sigma(z_{t+1})) + \frac{1 - \sigma(z_{t+1}) - \gamma\lambda_w(\theta_{t+1})}{1-\gamma}\frac{\kappa}{\lambda_f(\theta_{t+1})}\right]. \tag{20.29}$$

Equation (20.29) determines the dynamics of θ_t. Here, $\sigma(z)$ is determined in equilibrium by (20.28) (because θ is a function of z, σ is also a function of z).

20.7.2 Log-linearized system

Once again, we work with a log-linearized system. First, let the maintenance cost function be

$$c(\sigma) = \phi\sigma^{-\xi},$$

where $\phi > 0$ and $\xi > 0$ are parameters. Assume that the matching function takes the form (20.23). The derivation of the log-linearized system is similar to the exogenous separation case and detailed in Appendix 20.A.5.

First, guess the log-linearized relationship between θ_t and z_t as

$$\hat{\theta}_t = \mathcal{G}\hat{z}_t. \tag{20.30}$$

Then (20.28) can be log-linearized to

$$\hat{\sigma}(z_t) = -\frac{\eta}{\xi+1}\mathcal{G}\hat{z}_t. \tag{20.31}$$

Thus the log-deviation of the cost is (using the shortened notation of $c(z) = c(\sigma(z))$)

$$\hat{c}(z_t) = \frac{\xi\eta}{\xi+1}\mathcal{G}\hat{z}_t. \tag{20.32}$$

Using (20.31) and (20.32), the log-linearized version of (20.29) can be rewritten in terms of $\hat{z}_t$, $\hat{z}_{t+1}$, $\hat{\theta}_t$, and $\hat{\theta}_{t+1}$. Then, similarly to the exogenous separation case, the coefficient $\mathcal{G}$ can be solved as

$$\mathcal{G} = \frac{\Theta}{\Gamma},$$

where

$$\Theta \equiv (1-\gamma)\left[\frac{\kappa\bar{\theta}^{\eta}\eta}{\chi}\left(\frac{1}{\rho\beta} - (1-\bar{\sigma})\right) + \kappa\gamma\bar{\theta}\right]^{-1}\bar{z}$$

and

$$\Gamma \equiv 1 + (1-\gamma)\left[\frac{\kappa\bar{\theta}^{\eta}\eta}{\chi}\left(\frac{1}{\rho\beta} - (1-\bar{\sigma})\right) + \kappa\gamma\bar{\theta}\right]^{-1}\bar{c}\frac{\xi\eta}{\xi+1}\left(1 - \frac{1}{1-\gamma}\right).$$

where $\bar{\sigma}$ is the steady-state value of σ and $\bar{c} = \phi\bar{\sigma}^{-\xi}$ is the steady-state value of the maintenance cost.

20.7.3 Calibration and quantitative results

Parameters β, ρ, σ_{ε}, χ, b, γ, and η are set at the same value as in the previous section. For σ, we set the other parameters so that $\bar{\sigma} = 0.034$ matches the average separation rate in the U.S. data.

The newly-introduced specification is the maintenance cost function, $c(\sigma) = \phi\sigma^{-\xi}$. (20.30), (20.31), and $\lambda_w(\theta) = \chi\theta^{1-\eta}$ imply that the ratio of the standard deviations is

$$\frac{\text{std}(\hat{\lambda}_w)}{\text{std}(\hat{\sigma})} = \frac{(1-\eta)(1+\xi)}{\eta}.$$

Krusell et al. (2017, Table 8) indicates the ratio of the standard deviations for the *EU* flow rate and the *UE* flow rate is close to 1. Thus, for $\eta = 0.72$, we set $\xi = 1.6$.

As in Section 20.6.2, we set the steady-state value of θ as 1, and for a given κ, we can determine the steady-state value of ϕ from $\bar{\sigma} = 0.034$. Thus $c(\bar{\sigma})$ can be expressed in κ, and as in Section 20.6.2, we can set the value of κ from the steady-state version of (20.29).

The result is in Table 20.5. The fluctuation of u is still quantitatively very small compared to the data, although it is larger than the constant σ case, thanks to the movement in σ.

20.7.4 Rigid wages

Once again, we examine the situation where wages are rigid. Following the same steps as those in the generalized Nash bargaining case, it can be shown that

$$\frac{\kappa}{\lambda_f(\theta_t)} = \beta\mathbb{E}\left[z_{t+1} - \bar{w} - c(z_{t+1}) + \frac{(1-\sigma(z_{t+1}))\kappa}{\lambda_f(\theta_{t+1})}\right]$$

Table 20.5: Model statistics with endogenous σ

		u	v	v/u	z
Standard Deviation		0.010	0.011	0.021	0.013
Quarterly Autocorrelation		0.862	0.623	0.764	0.765
	u	1	-0.893	-0.969	-0.909
Correlation Matrix	v	—	1	0.976	0.957
	v/u	—	—	1	0.961
	z	—	—	—	1

characterizes the dynamics of θ_t. Log-linearizing this equation, we obtain

$$\hat{\theta}_t = \left[\frac{\kappa\bar{\theta}^\eta\eta}{\chi}\left(\frac{1}{\rho\beta} - (1-\bar{\sigma})\right)\right]^{-1}\hat{z}_t.$$

It turns out that the obtained outcome is identical to (20.27). The terms with endogenous $c(z_t)$ and $\sigma(z_t)$, which are new elements here, exactly cancel out. The log-linearized equations for the dynamics of $\sigma(z_t)$ and $c(z_t)$, (20.31) and (20.32), are the same as in the generalized Nash bargaining case.

The results are in Table 20.6. The model can replicate the large fluctuations in unemployment. In fact, the fluctuations in u are larger than in the data, suggesting that even a less extreme form of wage rigidity can generate substantial fluctuations in u in this case.

Table 20.6: Model statistics with endogenous σ and fixed wages

		u	v	v/u	z
Standard Deviation		0.217	0.232	0.433	0.013
Quarterly Autocorrelation		0.852	0.609	0.763	0.765
	u	1	-0.854	-0.960	-0.901
Correlation Matrix	v	—	1	0.966	0.948
	v/u	—	—	1	0.961
	z	—	—	—	1

20.8 Labor market frictions and the neoclassical growth model

As the final section of this chapter, we connect the DMP model to the main workhorse model of this book: the neoclassical growth model.[14] The important changes are (i) concave utility with an explicit consumption-saving problem and (ii) the use of capital (in addition to labor) in production. In addition, as discussed in the footnote 4 briefly, the earlier sections

[14] The model in this section follows Krusell, Mukoyama, and Şahin (2010, Appendix O). Earlier papers incorporating the search and matching framework into the neoclassical growth model include Merz (1995) and Andolfatto (1996).

implicitly assume that all firms are owned by someone outside the economy. In this section, we consider a closed economy, and therefore the profit income from the firm ownership is made explicit.

20.8.1 The baseline model with generalized Nash bargaining

Imagine that there are consumers on the unit square. The mass of consumers is one. The consumers are indexed by (i, j), where $i \in [0, 1]$ and $j \in [0, 1]$. The index i indicates the family the consumer belongs to. The family i, therefore, has members (indexed by j) on a unit interval. Families are identical to each other, and therefore, we will consider the *representative family*. Within each family, the members insure each other. That is, although some members are employed and others are unemployed, the income is pooled at the family level, and each member consumes the same amount (here, we do not consider disutility from work). Thus we have families that are homogeneous, and the family members are identical within each family. Below we will consider the decision of the representative family. Because the consumption of each family member is identical, the "family head" only needs to think about the representative member of the family.

Assume that each family member's utility is (because we consider the representative member of the representative family, we omit the indices i and j below)

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \mathbf{U}(c_t) \right],$$

where $\mathbb{E}_0[\cdot]$ is the expectation taken at time 0, c_t is and $\mathbf{U}(\cdot)$ is an increasing and concave period utility function.

The budget constraint for the family is

$$c_t + k_{t+1} = (1 + r_t - \delta)k_t + (1 - u_t)w_t + u_t b + d_t,$$

where k_t is the capital stock holding, r_t is the rental rate of capital, $\delta \in (0, 1]$ is the depreciation rate of capital, u_t is the unemployment rate, w_t is the wage per worker, b is the home production of unemployed workers, and d_t is the dividend from the firm.

The labor market setting is the same as in Section 20.4. In this section, we assume that production uses both capital and labor. Normalizing the labor input per match as 1, the output per match is assumed to be $z_t k_t^{\alpha}$, where $\alpha \in (0, 1)$. We assume that the capital is rented by firms from the families every period. Thus, the maximization problem for the choice of capital input by the firm is

$$\max_{k_{f,t}} \; z_t (k_{f,t})^{\alpha} - r_t k_{f,t}.$$

The optimal capital input satisfies

$$\alpha z_t (k_{f,t})^{\alpha-1} = r_t.$$

In equilibrium, there is a mass $(1 - u_t)$ of matches, which equally divides the total capital stock in the economy. Therefore, r_t in the equilibrium is

$$r(z_t, K_t, u_t) = \alpha z_t \left(\frac{K_t}{1 - u_t} \right)^{\alpha-1}.$$

Below, let us define $X_t \equiv (z_t, K_t, u_t)$. The surplus per match (denoted by z_t in Section 20.4) is now

$$y(X_t) \equiv z_t \left(\frac{K_t}{1-u_t}\right)^{\alpha} - r(X_t)\left(\frac{K_t}{1-u_t}\right) = (1-\alpha) z_t \left(\frac{K_t}{1-u_t}\right)^{\alpha}.$$

Firms also solve the dynamic problem of vacancy posting, as in the standard DMP model. Because the representative family's utility function is not linear, the discount factor can be different from β.

For the purpose of exposition, we divide the equilibrium of this model into two blocks: the consumption-saving block and the labor market block. The consumption-saving problem of the representative family can be written as the Bellman equation

$$\mathbf{V}(k, X) = \max_{c,k'} \; \mathbf{U}(c) + \beta \mathbb{E}[\mathbf{V}(k', X')|z] \tag{20.33}$$

subject to

$$c + k' = (1 + r(X) - \delta)k + (1-u)w(X) + ub + d(X),$$
$$K' = \Omega(X), \tag{20.34}$$

and

$$u' = (1 - \lambda_w(\theta(X)))u + \sigma(1-u), \tag{20.35}$$

where prime ($'$) represents the next period variable. The family takes the rental rate $r(X)$, the wage $w(X)$, the dividend $d(X)$, the law of motion for aggregate capital $\Omega(X)$, and the labor-market tightness $\theta(X)$ as given (as functions of X). Later on, we will confirm that $w(X)$, $d(X)$, $\Omega(X)$, and $\theta(X)$ are functions of X. From the solution to this Bellman equation, we obtain the decision rules $c(k, X)$ and $k'(k, X)$. Because the families are identical, the equilibrium aggregate consumption is $C(X) = c(K, X)$ and the next period aggregate capital is $\Omega(X) = k'(K, X)$. Note that (20.35) implies we can express u' as a function of X, $u'(X)$.

From this information, we can express the state price (i.e., the price of an Arrow security) of the next period state z' when the current state is X as

$$Q(z', X) = \beta f(z'|z) \frac{\mathbf{U}'(C(z', \Omega(X), u'(X)))}{\mathbf{U}'(C(X))}. \tag{20.36}$$

Here, $f(z'|z)$ is the probability density of state z' given the current state z and $u'(X)$ represents the right-hand side of (20.35). The derivation is in Appendix 20.A.6. In summary, once we know the functions $w(X)$, $d(X)$, and $\theta(X)$, we can obtain the state price $Q(z', X)$, in addition to other functions that include $\Omega(X)$ by solving the consumption-saving block. Below, we show that once we have $Q(z', X)$ and $\Omega(X)$, we can obtain $w(X)$, $d(X)$, and $\theta(X)$ in the "labor market block" below. Then these five functions $(Q(z', X), \Omega(X), w(X), d(X), \theta(X))$ can be computed as a fixed point.

Thus suppose we know $Q(z', X)$ and consider the labor market. It works very similarly to the basic DMP model. A firm with a worker has a value $J(X)$, where

$$J(X) = y(X) - w(X) + \int Q(z', X)[(1-\sigma)J(X') + \sigma V(X')]dz'. \tag{20.37}$$

Here, we discount the future value with $Q(z', X)$ because it represents the price of the next period good (as a function of z') in terms of the current good. The derivation of (20.37) (and the other asset value equations) can be found in Appendix 20.A.7. The value of vacancy is

$$V(X) = -\kappa + \int Q(z', X)[\lambda_f(\theta(X))J(X') + (1 - \lambda_f(\theta(X))V(X')]dz'.$$

Here, the transition equation (20.34) and (20.35) are given, and the functions $w(X)$ and $\theta(X)$ are a part of the unknowns in this block. The free-entry condition, $V(X) = 0$, implies

$$\frac{\kappa}{\lambda_f(\theta(X))} = \int Q(z', X)J(X')dz'. \tag{20.38}$$

This equation is analogous to (20.10) in the basic DMP model.

On the worker side, from the family's viewpoint, a worker brings in a stream of income with a stochastically changing employment state. Thus, we can compute the value of having a worker with specific status for a family using the standard asset pricing theory (the "Lucas tree" model). The value of an employed worker is

$$W(X) = w(X) + \int Q(z', X)[(1 - \sigma)W(X') + \sigma U(X')]dz' \tag{20.39}$$

and the value of an unemployed worker is

$$U(X) = b + \int Q(z', X)[\lambda_w(\theta(X))W(X') + (1 - \lambda_w(\theta(X)))U(X')]dz'. \tag{20.40}$$

Because $J(X) - V(X)$ and $W(X) - U(X)$ are both linear in w, with the same procedure as in Section 20.4, the generalized Nash bargaining implies

$$(1 - \gamma)(W(X) - U(X)) = \gamma(J(X) - V(X)). \tag{20.41}$$

The generalized Nash bargaining here implies that the wage is indeed a function of X.

From (20.37), (20.39), (20.40), and $V(X) = 0$,

$$W(X) - U(X) + J(X) = y(X) - b + \int Q(z', X)[(1 - \sigma - \lambda_w(\theta(X)))(W(X') - U(X')) + (1 - \sigma)J(X')]dz'$$

Using (20.41),

$$\frac{J(X)}{1 - \gamma} = y(X) - b + \int Q(z', X)J(X')\frac{1 - \sigma - \gamma\lambda_w(\theta(X))}{1 - \gamma}dz'. \tag{20.42}$$

Moving one period forward, multiplying $Q(z', X)$ on both sides, integrating over z', and using (20.38) yields the job creation condition:

$$\frac{\kappa}{(1 - \gamma)\lambda_f(\theta(X))} = \int Q(z', X)\left[y(X') - b + \frac{1 - \sigma - \gamma\lambda_w(\theta(X'))}{1 - \gamma}\frac{\kappa}{\lambda_f(\theta(X'))}\right]dz'. \tag{20.43}$$

This condition confirms that the equilibrium θ can indeed be written as a function of X. From (20.37), (20.38), and (20.42), the wage can be solved as

$$w(X) = \gamma(y(X) - b) + b + \gamma\theta(X)\kappa. \tag{20.44}$$

The dividend is all firms' profit minus the vacancy cost. The number of filled jobs is $(1-u)$, and each job creates $y(X) - w(X)$ units of profit. The vacancy cost is $\kappa v = \kappa\theta(X)u$ because $\theta(X) = v/u$. Thus

$$d(X) = (1-u)(y(X) - w(X)) - \kappa\theta(X)u, \tag{20.45}$$

and once again, we confirm that $d(X)$ is a function of X.

As with the standard RBC models, there are several alternative methods to compute the equilibrium. The first is, as in the previous sections, to log-linearize the equilibrium conditions and solve for the coefficients.

The second method is to treat the equilibrium conditions as functional equations. For example, one method that can be employed is to first make a guess on $Q(z', X)$, then use (20.43) to find the function $\theta(X)$ (one can use an iterative method—starting from $\theta(X)$ on the right-hand side to obtain $\theta(X)$ in the left-hand side, etc.). Then we can compute $w(X)$ and $d(X)$ from (20.44) and (20.45). Using this information, the representative family's problem (20.33) can be solved using the standard techniques to solve the neoclassical growth models. Finally, $Q(z', X)$ is updated with (20.36). We iterate this process until convergence. The following simulation follows this latter method of computation.

The details of calibration and computation are in Appendix 20.A.8. Calibration is similar to Section 20.6.2. The only difference from Table 20.1 is that, in this model, the steady-state value of $y(X)$ is not 1. Thus we adjust the value of b so that it is 0.4 times the steady-state value of $y(X)$. As in Section 20.6.2, κ is endogenously calibrated. The utility is assumed to be a log function $\mathbf{U}(c) = \log(c)$. The production function parameter $\alpha = 0.4$ as in the standard RBC model (Cooley and Prescott, 1995). The value of δ in Cooley and Prescott (1995) is 0.012 in quarterly frequency, and thus we set $\delta = 0.004$ at a monthly frequency.

Table 20.7: Model statistics with generalized Nash bargaining: labor market

		u	v	v/u	z
Standard Deviation		0.005	0.017	0.022	0.015
Quarterly Autocorrelation		0.819	0.688	0.755	0.763
Correlation Matrix	u	1	−0.831	−0.899	0.089
	v	—	1	0.991	−0.071
	v/u	—	—	1	−0.078
	z	—	—	—	1

Table 20.7 computes the labor market statistics as in the earlier sections. The results are overall in line with Table 20.3. The only noticeable difference is that $\text{corr}(z, u)$, $\text{corr}(z, v)$, and $\text{corr}(z, v/u)$ are close to zero (and the signs are different). The reason is that, in this section's model, the production per worker $y(X)$ is affected not only by z, but also by k and u. In fact, the correlations of u, v, and v/u with the labor productivity $y(X)$ are similar to those with z in Table 20.3.

Table 20.8: Model statistics with generalized Nash bargaining: business cycles

	Y	C	I	L	Y/L
Standard Deviation	0.014	0.003	0.059	0.0004	0.014
Correlation with Y	1	0.875	0.991	0.902	0.99992

Table 20.8 computes the standard business cycle statistics that are typically computed in the Real Business Cycle (RBC) literature. Similar to the labor market statistics, all variables are aggregated to quarterly frequency, logged, and HP-detrended (with the smoothing parameter $\lambda = 1,600$). The business cycle properties are overall similar to the standard RBC model: all C, I, L, and Y/L (here, L is computed as $1-u$) are strongly procyclical, I is more volatile than Y, and C is less volatile than Y. The only significant difference is that L fluctuates much less than Y. In this model, this outcome reflects the unemployment volatility puzzle in Section 20.6.

20.8.2 Rigid wages

Now consider the case with rigid wages. The consumer's problem is the same as the generalized Nash bargaining case, except that the wage is rigid. Different from Section 20.6.4, the output per worker $y(X)$ moves not only with z, but also with k and u. Because the movement of $y(X)$ is relatively large, it turns out that the flow profit for the firm sometimes becomes negative. To maintain a positive profit, this time we assume the wage to be

$$\tilde{w}(X) = \max\{\bar{w}, y(X)\}.$$

In our simulation, in the majority of the periods, the wage remains $\bar{w}$.

The Bellman equation is

$$\mathbf{V}(k, X) = \max_{c,k'} \ \mathbf{U}(c) + \beta \mathbb{E}[\mathbf{V}(k', X')|z] \tag{20.46}$$

subject to

$$c + k' = (1 + r(X) - \delta)k + (1-u)\tilde{w}(X) + ub + d(X),$$
$$K' = \Omega(X),$$
$$u' = (1 - \lambda_w(\theta(X)) + \sigma(1-u),$$

The state price $Q(z', X)$ can, again, be computed as (20.36). The Bellman equation for the matched job is

$$J(X) = y(X) - \tilde{w}(X) + \int Q(z', X)[(1-\sigma)J(X') + \sigma V(X')]dz'. \tag{20.47}$$

The free-entry condition remains the same as (20.38), and thus (20.47) can be rewritten as

$$\frac{\kappa}{\lambda_f(\theta(X))} = \int Q(z', X)\left[y(X') - \tilde{w}(X) + \frac{(1-\sigma)\kappa}{\lambda_f(\theta(X'))}\right] dz'. \tag{20.48}$$

Similar to (20.45), the dividend is

$$d(X) = (1-u)(y(X) - \tilde{w}(X)) - \kappa\theta(X)u, \tag{20.49}$$

The computation is similar to the generalized Nash bargaining case, except that now $\tilde{w}(X)$ does not move as much. First make a guess on $Q(z', X)$. Second, we can solve for $\theta(X)$ from (20.48). Third, $d(X)$ can be computed from (20.49). Using these functions, we can solve (20.46) and update $Q(z', X)$. These steps are repeated until $Q(z', X)$ converges.

Table 20.9: Model statistics with rigid wages: labor market

		u	v	v/u	z
Standard Deviation		0.083	0.269	0.339	0.015
Quarterly Autocorrelation		0.818	0.671	0.744	0.763
Correlation Matrix	u	1	−0.811	−0.886	0.094
	v	—	1	0.990	−0.076
	v/u	—	—	1	−0.083
	z	—	—	—	1

The model calibration is the same as in Section 20.8.1. Table 20.9 describes the labor market statistics for the rigid wage case. As in Section 20.6.4, the response of v (and therefore u) to the productivity shock is magnified by the rigid wage. Similar to Table 20.7, the correlations of u, v, and v/u with z are weak. Once again, this result comes from the fact that $y(X)$ also moves with k and u. When we compute the correlation of these variables with $y(X)$, the pattern of correlations is similar to the results in the basic model.

Table 20.10: Model statistics with rigid wages: business cycles

	Y	C	I	L	Y/L
Standard Deviation	0.017	0.003	0.060	0.007	0.011
Correlation with Y	1	0.792	0.989	0.898	0.964

Table 20.10 lists the business cycle statistics. The results are similar to Table 20.8 in the previous section with the exception that the standard deviation of L is one order of magnitude larger, reflecting the larger variability of unemployment.

20.9 Heterogeneity of jobs and the frictional wage dispersion

So far, we have assumed that jobs are homogeneous, and workers always accept an offered job. In this section, we introduce a model where job offers are heterogeneous. Some jobs pay more than other jobs, and the kind of jobs offered to the worker is stochastic. It is assumed that every period, an unemployed worker can receive only one job offer. After receiving the offer, she decides whether to accept it. The model in this section is called the McCall search model (McCall, 1970) or simply the search model. The search model focuses on the worker's decision of whether to accept a job offer or to continue searching. The model only considers the worker's choice of labor supply and the demand side of the labor market is taken as given.

The heterogeneity of job offers gives rise to wage dispersion. The labor market friction plays a crucial role; if there are no frictions, all workers will accept only the best (highest-paying) job. Even with labor market frictions, there are challenges in modeling a setting where firms actively offer different wage levels. A well-known example is called the *Diamond paradox*. Diamond (1971) considered a setting with search frictions and showed that, if the jobs are homogeneous and the worker has to leave the job to look for another job (and thus search is costly), all firms offer the worker's *reservation wage* (the lowest wage the worker would accept). It is called a paradox because this extreme result holds even when the cost of search is very small. First, it is easy to check that this outcome constitutes a Nash equilibrium: workers do not look for another job if they know all other firms are offering their reservation wage, and no firm would want to deviate as offering a higher wage does not make it more likely that the firm will hire a worker. Second, the Nash equilibrium is unique because, with any other wage offer distribution, the firm that offers the highest wage always has an incentive to lower the wage slightly without losing the worker. In this section, instead of considering firms' wage setting behavior explicitly, we assume the (exogenous) heterogeneity of wage offers. This type of firm behavior can be justified by relaxing Diamond's assumptions. For example, one can assume that the jobs are heterogeneous or workers can search on the job.

Formally, suppose that the worker is infinitely lived and the utility of the worker is linear:

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t c_t\right].$$

An unemployed worker earns $b > 0$ every period. This income can be interpreted as home production, unemployment insurance benefit, or the value of leisure. Each worker receives one job offer every period. Job offers differ in terms of the wage w. It is stochastic with distribution function $F(w)$, which has a lower bound of 0 and upper bound w^u. The Bellman equation for an unemployed worker is therefore written as

$$U = b + \beta\int_0^{w^u}\max\{W(w), U\}dF(w), \tag{20.50}$$

where U is the value of unemployment and $W(w)$ is the value of employment with wage w. The expression (20.50) can be rewritten as

$$(1-\beta)U = b + \beta\int_0^{w^u}\max\{W(w) - U, 0\}dF(w). \tag{20.51}$$

Every period, an employed worker faces the probability $\sigma \in (0, 1)$ of losing her job. Therefore, the Bellman equation for an employed worker is

$$W(w) = w + \beta[(1-\sigma)W(w) + \sigma U]. \tag{20.52}$$

Equation (20.52) can be rewritten as

$$W(w) = \frac{w + \beta\sigma U}{1 - \beta(1-\sigma)}. \tag{20.53}$$

From this expression, we can see that $W(w)$ is increasing in w and $W(0) = \beta\sigma U/(1-\beta(1-\sigma)) < U$. We assume that w^u is sufficiently large so that $W(w^u) > U$. Therefore, there exists a threshold w^* where $W(w^*) = U$, $W(w) > U$ for $w > w^*$, and $W(w) < U$ for $w < w^*$. In other words, the wage level w^* is the worker's *reservation wage*. The value of w^* characterizes the worker's choice in this model.

Plugging the expression (20.53) into (20.51) and using that $W(w) > U$ if and only if $w > w^*$,

$$(1-\beta)U = b + \frac{\beta}{1-\beta(1-\sigma)} \int_{w^*}^{w^u} [w - (1-\beta)U] dF(w). \tag{20.54}$$

Considering the expression (20.53) for $w = w^*$ and noting $W(w^*) = U$, we obtain

$$(1-\beta)U = w^*.$$

Thus (20.54) can be rewritten as

$$w^* = b + \frac{\beta}{1-\beta(1-\sigma)} \int_{w^*}^{w^u} [w - w^*] dF(w). \tag{20.55}$$

This equation solves the reservation wage w^*.

What can we learn from this model? First, let us consider the frequency of job acceptance. The worker only accepts the jobs that are better than w^*. Thus the *job finding probability* λ is

$$\lambda = 1 - F(w^*).$$

The job finding probability is decreasing in w^*: when workers are more selective, they find jobs less often.

We can also conduct various comparative statics to analyze how changes in parameters affect the reservation wage w^* (and therefore λ). Rewrite (20.55) as:

$$\mathbf{G}(w^*, \beta, \sigma, b) = 0,$$

where

$$\mathbf{G}(w^*, \beta, \sigma, b) \equiv w^* - b - \frac{\beta}{1-\beta(1-\sigma)} \int_{w^*}^{w^u} [w - w^*] dF(w).$$

It is straightforward to show that $\partial\mathbf{G}/\partial\beta < 0$, $\partial\mathbf{G}/\partial\sigma > 0$, and $\partial\mathbf{G}/\partial b < 0$. For w^*, because (using Leibnitz's rule)

$$\frac{\partial}{\partial w^*} \int_{w^*}^{w^u} [w - w^*] dF(w) = -[1 - F(w^*)],$$

$\partial\mathbf{G}(w^*, \beta, \sigma, b)/\partial w^* > 0$. From the implicit function theorem, w^* is increasing in β and b and decreasing in σ. Intuitively, the worker becomes choosier (w^* becomes larger) when β

increases because the future gain from a better job has a higher weight compared to the opportunity loss from missing the immediate job. An increase in b makes the unemployment state more attractive and induces workers to wait longer. A higher σ implies that even a good job won't last long, and thus it becomes less attractive to wait for a good job offer to arrive.

An interesting comparative-statics exercise with this class of model is to analyze the effect of changes in the wage offer distribution. First, consider the change in the average wage. To analyze the change in average, suppose that the wage offer is $w+\varepsilon$ instead of w above (with the same distribution for w), and how the change in ε changes the reservation wage $w^*+\varepsilon$ when evaluated at $\varepsilon = 0$. Now the $\mathbf{G}$ function is modified to

$$\mathbf{G}(w^*, \varepsilon) \equiv w^* + \varepsilon - b - \frac{\beta}{1-\beta(1-\sigma)} \int_{w^*}^{w^u} [(w+\varepsilon) - (w^*+\varepsilon)] dF(w).$$

$$\frac{\partial}{\partial \varepsilon} \mathbf{G}(w^*, \varepsilon) = 1$$

and (using Leibniz's rule, evaluated at $\varepsilon = 0$)

$$\frac{\partial}{\partial w^*} \mathbf{G}(w^*, \varepsilon) = 1 + \frac{\beta}{1-\beta(1-\sigma)} [1 - F(w^*)].$$

Thus

$$\frac{dw^*}{d\varepsilon} = -\frac{1-\beta(1-\sigma)}{1-\beta(1-\sigma)+\beta[1-F(w^*)]}.$$

The change in the reservation wage is, therefore,

$$\frac{d(w^*+\varepsilon)}{d\varepsilon} = \frac{dw^*}{d\varepsilon} + 1 = \frac{\beta[1-F(w^*)]}{1-\beta(1-\sigma)+\beta[1-F(w^*)]} \in (0,1).$$

Thus the reservation wage goes up, but not one-to-one. When the average wage offer goes up by one dollar, the reservation wage goes up by less than one dollar. This outcome arises because b is kept constant. Because b is the same, the relative attractiveness of the unemployment state (compared to working) goes down. This effect attenuates the effect of the change in the wage offer distribution.

Next, consider the dispersion of the wage offer distribution. To analyze the effect of dispersion, we first have to define the appropriate concept of the dispersion of wage offers in this context. Here, we introduce the concept of the *mean-preserving spread*. For a random variable x with the distribution function $F(x)$, we can construct a random variable $\tilde{x} \equiv x+z$ where z has a distribution function $H_x(z)$ and its mean is zero ($\int z dH_x(z) = 0$). Then, the mean of $x+z$ is x, and let us call the new distribution function $G(\tilde{x})$. Then we refer to $G(\cdot)$ as a mean-preserving spread of $F(\cdot)$. It can be shown (see Mas-Colell, Whinston, and Green, 1995, p. 198) that $G(\cdot)$ being a mean-preserving spread of $F(\cdot)$ is equivalent to

$$\int_0^x G(t)dt \geq \int_0^x F(t)dt \text{ for all } x. \tag{20.56}$$

Now let us rewrite the equation (20.55) as[15]

$$\begin{aligned} w^* &= b + \frac{\beta}{1-\beta(1-\sigma)}\left[\int_0^{w^u}[w-w^*]dF(w) - \int_0^{w^*}[w-w^*]dF(w)\right] \\ &= b + \frac{\beta}{1-\beta(1-\sigma)}\left[\mu_w - w^* - \int_0^{w^*}[w-w^*]dF(w)\right], \end{aligned} \tag{20.57}$$

where μ_w is the mean value of w. Integration by parts yields

$$\int_0^{w^*}[w-w^*]dF(w) = -\int_0^{w^*}F(w)dw$$

and thus (20.57) can be rewritten as

$$w^* - b - \frac{\beta}{1-\beta(1-\sigma)}\left[\mu_w - w^* + \int_0^{w^*}F(w)dw\right] = 0. \tag{20.58}$$

It is straightforward to show that the left-hand side is increasing in w^*. Suppose that the distribution of w, $F(w)$, becomes more dispersed in the sense of the mean-preserving spread. The property (20.56) implies that the left-hand side of (20.58) goes down with this change in the distribution, and thus w^* has to go up. The reservation wage increases with the dispersion of the wage offers. Intuitively, the worker becomes choosier with more dispersed wage offers because the possibility of a good wage offer increases. With higher dispersion, the possibility of a bad offer also increases, but the left tail of the distribution does not matter because these offers are rejected in any case. In other words, the option value of searching increases with the dispersion of the wage offer.[16]

Now, consider the model implications for the realized wage dispersion. The equilibrium wage dispersion in this model is often called the *frictional wage dispersion* because all workers would work at $w = w^u$ if there are no search frictions (i.e., if all jobs are available to the workers). Workers accept a job with $w < w^u$ because it is costly to wait for high-wage job offers. To analyze the frictional wage dispersion, first define the mean (accepted) wage as

$$w^M \equiv \frac{\int_{w^*}^{w^u} w dF(w)}{1-F(w^*)}.$$

Let us also define

$$\rho \equiv \frac{b}{w^M},$$

that is, the ratio of the unemployed worker's income to the mean wage. When b is interpreted as the income from unemployment insurance, ρ corresponds to the replacement rate. Further, as defined above, let $\lambda \equiv 1 - F(w^*)$ be the job-finding probability, that is, the probability that an unemployed worker transitions into employment.

Define the *mean-min ratio* of the (accepted) wages, Mm, as

$$Mm \equiv \frac{w^M}{w^*}.$$

[15]This derivation follows Ljungqvist and Sargent (2012, pp. 166-167).
[16]See Mukoyama and Şahin (2009) for an application.

Note that w^* is the minimum value of the accepted wages. Then, using (20.55) and the above definitions, we can derive

$$Mm = \frac{1 + \beta\lambda/(1 - \beta(1 - \sigma))}{\rho + \beta\lambda/(1 - \beta(1 - \sigma))}.$$

A back-of-the-envelope calculation with (monthly) $\beta = 0.996$, $\sigma = 0.034$, $\lambda = 0.45$, and $b = 0.4$ yields $Mm = 1.031$. The mean wage is only 3.1% larger than the lowest wage in this economy. In other words, search frictions can explain only a tiny amount of wage dispersion in this model. This result is often referred to as the *frictional wage dispersion puzzle* in the literature.[17] The reason is that, in the model with this parameterization, it is not very costly to wait for a new job (the offer comes fairly frequently), whereas the benefit of receiving a better wage offer is large. One situation where the frictional wage dispersion is high is, therefore, when the cost of unemployment is high. A large cost of unemployment makes unemployed workers accept low wage offers to avoid unemployment. Another situation is when workers can search on the job. For example, if the offered wage distributions are identical and the offer frequency is the same for on- and off-the-job search, the worker would accept any job with $w > b$ when unemployed. In this case, the worker does not have to give up the option value of search when accepting a wage offer.

[17]See Hornstein, Krusell, and Violante (2011) for a detailed analysis.