Chapter 5

Dynamic competitive equilibrium

Chapter 2 argued that understanding macroeconomic history requires a particular view of production and technological change. It also suggested, informally at least, that market economies, in which firms and households make decentralized decisions in their own self-interest, play a central role in shaping aggregate outcomes. The present chapter formalizes that idea using the most widely adopted framework in economics: a perfectly competitive market setting. We develop this framework carefully in a dynamic context, setting the stage for the general equilibrium models used throughout the rest of the book.

Like many of our assumptions, perfect competition should be viewed as a useful approximation rather than an exact description of our economy. Indeed, most firms possess some market power, which in certain industries is essential for understanding their structure and behavior. However, a macroeconomic approach naturally starts from a benchmark case that is (i) not too far from a description of "average" behavior and (ii) allows tractable analysis, while still capturing the key features of markets. These features involve how private incentives—of consumers and firms—jointly steer production and consumption through a price mechanism. Market scarcity—expressed as a demand that exceeds supply—will lead to higher prices, lowering demand and/or raising supply. For example, investment—a key macroeconomic variable—is influenced by the interest rate, which reflects the relative price of goods today versus goods in the future. Consumers are more willing to forgo current consumption and lend to investors when the interest rate is high. In turn, investors continue to invest until the cost of borrowing equals the expected return on investment. This market mechanism, moreover, is at play not just for traditional consumption and investment goods and services but for the development of new ideas and technologies that are key to long-run development.

How do agents decide what to buy or sell? They solve an optimization problem that is—in a macroeconomic context—often dynamic, as we have seen in Chapter 4. The solution to the optimization problem provides the quantities of the objects that the agent would like to buy and/or sell at given prices; for example, a household could sell labor services and get paid a wage (the price for labor). Therefore, an important component of a competitive equilibrium is the characterization of individual demands and supplies for goods, services and assets, *given* the corresponding prices.[1] Since in the economy there are many agents who buy

[1] The notion that agents take prices as given allows us to analyze the economy without game-theoretic

and sell the traded objects, the derivation of economy-wide demands and supplies requires the aggregation of demands and supplies over all agents. The core framework described in this chapter makes this procedure simpler by assuming that all consumers are identical and that all firms are identical—we use the "representative consumer" and the "representative firm" as our key constructs. A large part of the current macroeconomic research literature of course focuses on consumer and firm heterogeneity, which will be amply discussed later in the text.

In summary, a perfectly competitive equilibrium occurs when two conditions are met: agents' choices are optimal taking the prices as given and markets clear, i.e., demand equals supply, for all goods and services. In this chapter, we will also see that the market allocation delivers a Pareto-optimal outcome. With a representative consumer, this amounts to verifying that, in the perfectly competitive equilibrium, the consumer's utility is maximized given all technological constraints. When we depart from this setting, and when monopoly elements and other "frictions" are introduced, markets need no longer deliver optimal outcomes. We look at such instances in some detail in Chapter 6 but they will then appear again and again in the applied chapters later.

5.1 Different equilibrium concepts

When considering our macroeconomic models, it is very helpful to be precise in formulating equilibrium concepts. A model economy, first, has some descriptive "deep" features, such as a set of agents (consumers, firms, a government, etc.) along with their objectives (such as utility functions) and relevant constraints (time available, technologies, etc.). Second, what the set of traded goods and services is—along with their prices—must be specified. The equilibrium concept then lists a set of conditions that need to be fulfilled: (i) agents maximizing their objectives subject to their constraints (e.g., consumers maximizing utility subject to their budget constraints and firms maximizing profits taking their technological possibilities into account) and (ii) resource feasibility: that what is consumed is also produced, often expressed as "demand equals supply," for all traded goods and services. The definition of an equilibrium organizes this information in the form of a set of conditions that need to be fulfilled. In this chapter, we formulate equilibrium definitions in a rather mathematical way, i.e., with a minimum number of words. As a result, we try to refrain from conditions such as "taking prices as given" in a maximization problem; rather, if a maximization problem is specified, it must list the choice variables, and it then follows that any variables that are not listed as choices (such as prices in a perfectly competitive equilibrium) must, then, be taken as given.

Formulating equilibria in dynamic models, moreover, involves a choice regarding how time is dealt with. We will, in particular, consider three different definitions of competitive equilibria in turn:

1. Arrow-Debreu equilibrium in which trades occur at date 0,

2. sequential equilibrium in which trades occur period by period, and

elements: in a given agent's decision problem, the behavior of others appears in the form of (endogenous) constants.

3. recursive equilibrium in which prices and decisions are expressed as functions of the economy's state variables.

In the first two of these, an equilibrium consists of sequences (of quantities and prices) indexed by time, i.e., a specific values for each variable at each point in time. The third concept, in contrast, formulates the equilibrium as a set of functions of state variables (such as a capital stock or level of asset holdings). For many economies, one can define the competitive equilibrium in any of these three ways and they all give rise to the same allocations. Which one is chosen in a given application depends on the purpose, but, broadly speaking, the Arrow-Debreu equilibrium is the version that aligns most closely with microeconomic theory while being rather abstract as a concept. Sequential equilibria are defined more in accordance with how we think events actually play out in reality. A recursive equilibrium has the sequential trading structure but is expressed in terms of functions rather than sequences because it uses dynamic programming methods. Whereas equilibria based on sequences typically are defined given specific values of the state variables (such as the initial capital stock), recursive equilibria are not, i.e., they apply to all values of the state variables.

Throughout, we will focus on dynamic models with an infinite time horizon. Finite-horizon economies can of course be studied too; they should be seen as special cases of what we study here. As pointed out above, infinite-horizon economies are useful because there, time plays a less central role: the remaining time horizon is always the same. Occasionally (later in the text), we focus on one-period (where the "time horizon" plays no role) or two-period economies, but later in the text. In Sections 5.2–5.4 we study infinitely-lived dynasties and go through the different equilibrium definitions. In the final part of the chapter, Section 5.5, we study overlapping-generations (OG) economies, where time goes on forever but all individuals live a finite number of periods.

5.2 Arrow-Debreu equilibrium

In an Arrow-Debreu equilibrium, all trades take place at time zero. One way to think about this assumption is that every agent signs a contract at time zero with other agents. The contract specifies the quantities of the traded objects that the agent will deliver or receive at any future time t from other agents at the prices specified. This market structure is called an Arrow-Debreu or date-0 market. It differs from the sequential market structure where trades for goods take place at all times t, not only at time zero. This different market structure will be considered in the other two equilibrium concepts.

It is perhaps worth emphasizing that signed contracts are fully enforceable, that is, promises made in a contract are always fulfilled. If some of the promises are not enforceable, we are in an environment in which markets are incomplete. We will consider economies with incomplete markets in later chapters.

The best way to illustrate the concept of an Arrow-Debreu equilibrium is through the application to specific economies. We start with an endowment economy.

5.2.1 An endowment economy

The first example is an economy in which production is exogenously determined. There is a continuum of consumers, indexed by $i \in [0,1]$. In every period t each consumer produces $y_{i,t} \in \mathbb{R}_+$ of a single consumption good that cannot be stored for future consumption. The model does not specify how production takes place (for example with the input of labor) and the exogenous production is often referred to as endowment. An economy without the specification of a production technology is sometimes called an *exchange economy*, since the only economic activity that agents undertake, besides consumption, is the trade of the endowment. But, effectively, the exchange economy can be seen as a special case of a production economy. Finally, the consumer's utility from any given consumption path $\{c_{i,t}\}_{t=0}^{\infty}$ is

$$\sum_{t=0}^{\infty} \beta^t u\left(c_{i,t}\right). \tag{5.1}$$

Consumers' preferences are all the same (they have the same u and β).

What is traded here is goods at different points in time: buying one unit of good at time t means buying a contract that gives ownership of a unit of good at that point in time. Its price is denoted by p_t. Thus, p_t/p_0 represents the units of consumption goods delivered at time 0 that are needed to buy 1 unit of the consumption good delivered at time t. It is customary to normalize the price at time zero to 1 so that the good at time 0 becomes the numéraire. Then p_t is the price of a time-t good in terms of time-0 consumption goods.

Given the price p_t for $t = 0, 1, \ldots$, the total value of the consumer's endowments is given by $\sum_{t=0}^{\infty} p_t y_{i,t}$ and the value of the consumer's total expenditures is $\sum_{t=0}^{\infty} p_t c_{i,t}$. The budget constraint then requires that the value of expenditures not be larger than the value of the endowments. In practice, since we always use strictly increasing utility functions, consumers will always use up all the resources and we will therefore impose this constraint with equality. We have

$$\sum_{t=0}^{\infty} p_t c_{i,t} = \sum_{t=0}^{\infty} p_t y_{i,t}. \tag{5.2}$$

We will define the equilibrium focusing on the mathematical conditions that must be satisfied, as opposed to their economic interpretation. The equilibrium definition provides a roadmap for solving for an equilibrium—we convert the equilibrium conditions into a set of equations that allow us to solve the model.[2]

Definition 1 *An* ***Arrow-Debreu competitive equilibrium*** *is a set of sequences* $\left\{c_{i,t}^{\star}\right\}_{t=0}^{\infty}$*, for each* $i \in [0,1]$*, and* $\{p_t\}_{t=0}^{\infty}$ *such that*

1. *for each* i*,* $\left\{c_{i,t}^{\star}\right\}_{t=0}^{\infty}$ *solves*

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \textit{subject to} \quad \sum_{t=0}^{\infty} p_t c_t = \sum_{t=0}^{\infty} p_t y_{i,t};$$

[2]Here and in what follows, we omit any requirements that quantities and prices be non-negative. After we solve for an equilibrium we can verify that any such requirements are satisfied. Also, we do not specify the mathematical spaces to which the sequences must belong; for infinite sequences, this involves advanced concepts, which is why we omit them.

2. for each t, $\int_0^1 c^\star_{i,t} di = \int_0^1 y_{i,t} di$.

Equilibrium characterization Let us now characterize equilibrium. We will gradually introduce more detail for our assumptions regarding the utility function and endowment sequences. To characterize the optimal decision of a consumer (point 1 in the above definition), we write the Langragian for the consumer problem. We then take the first-order conditions as shown in Chapter 4. This gives us

$$\beta^t u'(c_{i,t}) = \lambda_i p_t,$$

where λ_i denotes the Lagrange multiplier for the (lifetime) budget constraint, equation (5.2), and the prime denotes the derivative of the utility function. The multiplier is the shadow value of lifetime wealth which could differ across agents if their lifetime endowments differ. However, the multiplier does not depend on time. If the utility function is strictly concave, which we will assume here, its derivative is strictly decreasing in consumption. Thus, the above condition determines, uniquely, the optimal consumption for agent i.

If we eliminate the multiplier using the optimality conditions at time zero and at a generic time t, for the same consumer, we obtain the price

$$p_t = \beta^t \frac{u'(c_{i,t})}{u'(c_{i,0})}. \tag{5.3}$$

Remember that p_0 has been normalized to 1, which explains why it does not appear in this expression. Since the price p_t is the same for all agents, this condition tells us that the ratio of marginal utilities at different times are the same across consumers. This equation states that the relative price of consumption at time t in terms of time zero consumption has to equal the marginal rate of substitution between these two goods. This is the ratio between the present value of the marginal utility of consumption at time t and the marginal utility of consumption at time zero.

To gain further intuition, we now consider the special case in which the utility function takes the logarithmic form, that is, $u(\cdot) = \log(\cdot)$. With this special utility, the above condition becomes

$$p_t = \beta^t \frac{c_{i,0}}{c_{i,t}}.$$

Since this condition must hold for all agents, we can use this condition to find equilibrium prices. Multiply both sides of the equation by $c_{i,t}$ and sum across all i. We obtain $p_t \int_0^1 c_{i,t} di = \beta^t \int_0^1 c_{i,0} di$. Using market clearing, i.e., $\int_0^1 c_{i,t} di = \int_0^1 y_{i,t} di = Y_t$, we obtain

$$p_t = \beta^t \frac{Y_0}{Y_t}.$$

Thus, we see that prices decline over time because of discounting but also that periods with lower total endowments have higher prices. This is because marginal utility is strictly decreasing: lower total resources make each unit of consumption more valuable on the margin.

From the Euler equation, we see that in this economy all consumers experience the same consumption growth. More specifically, if we take any two consumers, i and i', we have

$$\frac{c_{i,t+1}}{c_{i,t}} = \frac{c_{i',t+1}}{c_{i',t}} = \frac{Y_{t+1}}{Y_t} = \beta \frac{p_t}{p_{t+1}}.$$

This condition is valid for any pattern of the individual endowments. Clearly, the growth rate of individual consumption is only determined by the aggregate endowment Y_t. Therefore, individual consumption could be less volatile than individual income (consumption smoothing). To see this more clearly, suppose for a moment that individual endowments change over time but that the aggregate endowment, $Y_t = \bar{Y}$, does not. Therefore, $\int_0^1 y_{i,t} di = \bar{Y}$. In equilibrium it must be that $\int_0^1 c_{i,t} di = \bar{Y}$. In other words, aggregate consumption is constant. Since the consumption growth of all agents is the same, this implies that individual consumption must also be constant for all $t = 0, 1, 2, \ldots$. Therefore, in this special case with a constant aggregate endowment, but not necessarily constant individual endowments, the model features perfect consumption smoothing.

Lastly, the level of consumption for each consumer can be derived from the individual budget constraint. By combining the Euler equations for $t = 0, 1, 2, ...$ and $p_0 = 1$ we arrive at $p_t c_{i,t} = \beta^t c_{i,0}$. The budget constraint then becomes

$$c_{i,0} \sum_{t=0}^{\infty} \beta^t = Y_0 \sum_{t=0}^{\infty} \beta^t \frac{y_{i,t}}{Y_t} \quad \Rightarrow \quad c_{i,0} = (1-\beta) Y_0 \sum_{t=0}^{\infty} \beta^t \frac{y_{i,t}}{Y_t}.$$

Clearly, the level of consumption of individual i, as measured by its level at time 0, is a fraction $1 - \beta$ of the individual's present-value income. This income is a function of endowments where (i) endowments further into the future obtain lower weights due to discounting and (ii) endowments in periods where aggregate income is high obtain a lower weight. The latter is true since resources in periods with high aggregate income are given lower value by consumers: their marginal utilities are lower then than in other periods. Hence, two consumers with the same average endowments can have different total wealth: the wealthy consumer is the one whose endowments are high when others' endowments are low.

Clearly, since consumption growth is identical for all consumers, we see that individual i's share of aggregate consumption, and aggregate resources, is constant over time:

$$c_{i,t} = \theta_i C_t = \theta_i Y_t, \quad \text{with} \quad \theta_i = (1-\beta) \sum_{t=0}^{\infty} \beta^t \frac{y_{i,t}}{Y_t}.$$

In the special case in which agents have exactly the same endowment ($y_{i,t} = Y_t$), the share is 1 for all agents which effectively means that they consume their own endowment. We are then in the case of a *representative consumer*.

If the utility function is not logarithmic but maintains the balanced-growth form $(c^{1-\sigma} - 1)/(1-\sigma)$ (with $\sigma > 0$ and $\sigma \neq 1$; recall that $\sigma \to 1$ should be interpreted as $\log c$), then it is still possible to solve the model. It is easy to verify, using the consumer's Euler equation, that individuals' consumption growth rates will all be identical: the gross rates will equal $(\beta p_t / p_{t+1})^{\frac{1}{\sigma}}$. From this it follows that $p_t = \beta^t (Y_0/Y_t)^{\sigma}$: again, resources are more valuable in times with lower endowments, and the more so the higher is the curvature of the utility function, as measured by σ.

The endowment framework studied here is useful in many contexts. One of these is asset pricing, covered in Chapter 16. The idea there is that any asset can be thought of as a stream of payments, "dividends," such as the endowment sequences described here, so the price of the asset is then the total market value of these endowments. Prices of endowments at each time period are straightforward to compute in endowment economies, including in cases with uncertainty; we will look at uncertainty in Chapter 7.

Finally, in many macroeconomic applications, the assumption of a representative agent is used. It makes sense as a special case of the above, when u is strictly concave: then, given equal endowments for all agents, the consumption choices must all be the same. Such an equilibrium is often defined more compactly as

Definition 2 *An* ***Arrow-Debreu competitive equilibrium*** *is a set of sequences* $\{c_t^\star\}_{t=0}^\infty$ *and* $\{p_t\}_{t=0}^\infty$ *such that*

1. $\{c_t^\star\}_{t=0}^\infty$ *solves*

$$\max_{\{c_t\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t u(c_t) \quad \textit{subject to} \quad \sum_{t=0}^\infty p_t c_t = \sum_{t=0}^\infty p_t y_t;$$

2. $c_t^\star = y_t$.

This equilibrium definition is mathematically precise but the economic context—that of many consumers, making the same choices—is only written "between the lines."

5.2.2 A production economy with labor

We now consider an economy where there is production using only labor as the input. The economy is populated by a continuum of households, each supplying working hours $\ell_{i,t}$ to the market. However, working is costly for the household as it reduces utility. Given the consumption path $\{c_{i,t}\}_{t=0}^\infty$ and working hours $\{\ell_{i,t}\}_{t=0}^\infty$, the household's utility is

$$\sum_{t=0}^\infty \beta^t \left[u\left(c_{i,t}\right) - v(\ell_{i,t})\right]. \tag{5.4}$$

The function $v(\ell_{i,t})$ is the disutility from working. It is increasing and convex in $\ell_{i,t}$. We allow households to differ in their efficiency units of labor which we denote by e_i. What this means is that, if household i works $\ell_{i,t}$ hours, its contribution to the economy's input of labor (in efficiency units) is $e_i\ell_{i,t}$. We can think of e_i as labor skills.

In this economy there is also a representative firm that produces consumption goods with the production function

$$Y_t = A_t L_t, \tag{5.5}$$

where A_t could be time varying but not stochastic, and L_t is the effective input of labor used in production.

In a competitive economy, the presumption is also that there are many firms. However, if these firms all have the same objective (profit maximization) and the same technology available, then they will face identical maximization problems. For this reason, we will use the concept of a *representative firm*. It hires labor from households paying the wage rate w_t per effective unit of labor in terms of the consumption good at that time. Therefore, if the household supplies one efficiency unit of labor at time t, it receives w_t units of consumption goods at that time, which translates to $p_t w_t$ units in terms of time-0 good.

The representative firm's objective is to maximize profits:

$$\max_{\{L_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \Big\{ p_t A_t L_t - p_t w_t L_t \Big\}. \tag{5.6}$$

Clearly, L_t only appears in the time-t term, so this problem reduces to an infinite sequence of static problems $\max_{L_t}(A_t L_t - w_t L_t)$; the within-period relative price w_t is the price that will clear the time-t market. Unlike for the concept of the representative consumer, whose optimal choice is unique, the optimal firm choice is—under constant returns to scale and price-taking—not unique in equilibrium: the firms is indifferent as to what scale to operate at.[3]

Given prices p_t and $p_t w_t$ and labor supplies $\ell_{i,t}$, for $t = 0, 1, ...$, the value of lifetime income for household i is $\sum_{t=0}^{\infty} p_t w_t e_i \ell_{i,t}$ and the value of its expenditures is $\sum_{t=0}^{\infty} p_t c_{i,t}$. The budget constraint requires that the value of expenditures not be larger than the value of incomes, that is,

$$\sum_{t=0}^{\infty} p_t c_{i,t} = \sum_{t=0}^{\infty} p_t w_t e_i \ell_{i,t}. \tag{5.7}$$

The following is then the compact definition of our equilibrium.

Definition 3 *An* ***Arrow-Debreu competitive equilibrium*** *is a set of sequences* $\left\{c^\star_{i,t}\right\}_{t=0}^{\infty}$ *and* $\left\{\ell^\star_{i,t}\right\}_{t=0}^{\infty}$*, for each* $i \in [0,1]$*,* $\{L^\star_t\}_{t=0}^{\infty}$*,* $\{p_t\}_{t=0}^{\infty}$ *and* $\{w_t\}_{t=0}^{\infty}$ *such that*

1. *for each* i*,* $\left\{c^\star_{i,t}\right\}_{t=0}^{\infty}$ *and* $\left\{\ell^\star_{i,t}\right\}_{t=0}^{\infty}$ *solve*

$$\max_{\{c_t, \ell_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[u(c_t) - v(\ell_t)\right] \quad \textit{subject to} \quad \sum_{t=0}^{\infty} p_t c_t = \sum_{t=0}^{\infty} p_t w_t e_i \ell_t;$$

2. $\{L^\star_t\}_{t=0}^{\infty}$ *solves*

$$\max_{\{L_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \Big\{ p_t A_t L_t - p_t w_t L_t \Big\};$$

3. *for each* t*,* $\int_0^1 e_i \ell^\star_{i,t} di = L^\star_t$ *and* $\int_0^1 c^\star_{i,t} di = A_t L^\star_t$ *.*

The first two points say that the allocation is what agents (households and firms) choose optimally to maximize their respective objective functions.[4] The third item defines the market clearing condition for labor (the aggregate supply of labor from households must be equal to the aggregate demand from firms) and for the goods markets (the aggregate demand of goods from households must be equal to the aggregate supply, which corresponds to production).

[3]In this case, an equilibrium price for labor, w_t, has to equal A_t, in which case the firm is indifferent as to its choice of L_t; if $A_t > w_t$, there is no optimal scale (profits rise without bound with scale), and if $A_t < w_t$, the firm's unique maximizer is $L_t = 0$.

[4]For the firm, we have written it so that the firm chooses labor and production at all times. Since the problem is not fundamentally dynamic—L_t can be chosen independently of labor chosen at times other than t—we could alternatively had firms operating production at single dates only.

Equilibrium characterization The optimality conditions for the consumer's problem (again derived from the Lagrangian as shown in Chapter 4) are

$$\beta^t u'(c_{i,t}) = \lambda_i p_t, \tag{5.8}$$

and

$$v'(\ell_{i,t}) = e_i w_t u'(c_t), \tag{5.9}$$

where λ_i is the Lagrange multiplier for the (lifetime) budget constraint, equation (5.7).

We can eliminate the multiplier using the optimality conditions for an agent i, at time t and at time $t+1$, and obtain

$$\frac{p_{t+1}}{p_t} = \frac{\beta u'(c_{i,t+1})}{u'(c_{i,t})}. \tag{5.10}$$

As in the case of the endowment economy we just studied, the relative price of tomorrow's consumption in terms of today's consumption—the inverse of the (gross) real interest rate—equals the marginal rate of substitution between consumption at t and $t+1$. With the normalization $p_0 = 1$, the equilibrium price at any time t is

$$p_t = \frac{\beta^t u'(c_{i,t})}{u'(c_{i,0})}.$$

The firm solves (5.6) and the first-order condition requires that, for an interior solution—where the firm is indifferent between using more or less labor—we need to have

$$A_t = w_t.$$

If A_t were to exceed w_t, the firm's maximization problem would have no solution: the higher is L_t, the better. If w_t were to exceed A_t the firm would shut down (choose zero labor). Thus, for an equilibrium to exist, the relative price w_t needs to adjust to be equal to A_t. When it has, the firm's size is indeterminate: all $L_t \geq 0$ deliver zero profits. This is a standard result under constant returns to scale and will apply also when the firm has more inputs (such as labor an capital, as in the neoclassical growth model). Our notation may suggest that there is one firm that produces the economy's entire output but, really, we consider a large number of firms solving the same problem under perfect competition, and the equilibrium will then determine total production but not which firm produces how much.

We now again consider the special case in which the utility function takes the logarithmic form, that is, $u(\cdot) = \log(\cdot)$. With this utility equation (5.10) can be written as

$$\frac{c_{i,t+1}}{c_{i,t}} = \frac{\beta p_t}{p_{t+1}}.$$

Since prices are the same for all agents, this implies that households experience the same growth in consumption. Thus, also in this case we have that individual consumption is a constant share of aggregate output, that is, $c_{i,t} = \theta_i Y_t$. However, output is not exogenous but it depends on the endogenous input of labor L_t. This is determined by the aggregation of individual labor supplies which are determined by the first-order condition (5.9). Using the log specification of the utility function and the fact that the wage rate is given by $w_t = A_t$, the condition can be rewritten as

$$v'(\ell_{i,t})c_{i,t} = e_i A_t.$$

Next we use the property that individual consumption is a fixed share of aggregate output $c_{i,t} = \theta_i Y_t$. Substituting in the first-order condition we obtain

$$v'(\ell_{i,t})\theta_i Y_t = e_i A_t.$$

Let us normalize skills so that $\int_0^1 e_i di = 1$. Given that skills are constant over time, different consumers have different total resources to spend in exact proportion to e_i, aside from differences in how much they work. However, we see that if we conjecture $\theta_i = e_i$, then the above condition implies that all households will supply the same labor $\ell_{i,t} = L_t$, confirming that total resources available to spend are simply proportional to e_i.

To see that the lifetime budget (5.7) is satisfied, use $c_{i,t} = \theta_i Y_t$, $w_t = A_t$, and $\ell_{i,t} = L_t$ in the budget constraint to obtain

$$\theta_i \sum_{t=0}^{\infty} p_t A_t L_t = e_i \sum_{t=0}^{\infty} p_t A_t L_t.$$

This is satisfied if $\theta_i = e_i$, confirming our guess.

Because high-skilled households earn higher incomes, they will enjoy higher consumption. High-skilled households, however, supply the same amount of labor as do low-skilled households. This is a consequence of income and substitution effects offsetting each other. To see this, note that an extra unit of work effort earns $e_i w_t$, which increases with the wage and the skill of the worker. This generates a substitution effect in the direction of working more. However, since—without working more—the household earns higher consumption the higher is the wage, it experiences a lower marginal utility of consumption, which leads it to wish to work less (choose lower effort/higher leisure). With the utility function here, the two effects exactly cancel. These features play out even more clearly in a static optimization problem where the consumer has labor income only and spends it on consumption, $c = w\ell$, with the utility function $\log c - v(\ell)$: the choice of ℓ will not depend on w. This feature was alluded to in Chapter 2, where we argued that balanced growth with constant labor supply restricts us to a specific class of utility functions, to which the current example belongs.

Finally, imagine that individuals, in addition to being able to work, had independent ("asset") income. In particular, individual i would be endowed with $a_{i,0}$ at time zero, with $\int_0^1 a_{i,0} di = 0$. This assumption implies that, if some agents have positive asset holdings, other agents must have negative holdings; therefore, the resource constraints remain unchanged. Now, agents with different asset holdings will work different amounts because assets generate an income effect but not a substitution effect: higher asset holdings, by making the individual richer, will induce lower labor effort. Consider the simple static case just described: if the budget constraint reads $c_i = e_i A \ell_i + a_i$, individual labor supply will, using the first-order condition, be given by $v'(\ell_i) = a_i A/(e_i A \ell_i + a_i)$. Here, ℓ_i will depend nonlinearly on a_i (so long as v' is not a constant). Hence, total labor supply, along with total consumption, will depend on the distribution of assets: they will depend on wealth inequality. Consumption growth would still be equalized across agents, but total production will also depend on inequality since total consumption does.

5.2.3 The neoclassical growth economy

We extend the economy considered in the previous subsection by adding capital to production. This is essentially the neoclassical growth model with endogenous supply of labor. The accumulation of capital makes the problem more complex because what is produced in the future depends on the capital that is accumulated today. In most cases, a full analytical solution will not be available. Nevertheless, we can define the conditions that an equilibrium must satisfy.

The production function now takes the form

$$Y_t = A_t K_t^\alpha L_t^{1-\alpha}, \tag{5.11}$$

where A_t is productivity, K_t is the input of capital and L_t is the input of labor.

We assume that capital is accumulated by households who then rent it to firms, similarly to labor. Therefore, in addition to the price for goods and labor, we now have a price for rental capital. The representative firm solves the profit maximization problem

$$\max_{\{K_t,L_t\}_{t=0}^\infty} \sum_{t=0}^\infty \left\{ p_t A_t K_t^\alpha L_t^{1-\alpha} - p_t r_t K_t - p_t w_t L_t \right\}, \tag{5.12}$$

where r_t denotes the rental price of capital paid to households expressed in time-t consumption. As in the previous sub-section, the optimal choice of the firm for period t does not affect the optimal choices in other periods; we can equivalently state the problem as a sequence of sub-problems: for each t, $\max_{K_t,L_t} A_t K_t^\alpha L_t^{1-\alpha} - r_t K_t - w_t L_t$.

In this economy, production is used for consumption, C_t, and for investment, I_t:

$$Y_t = C_t + I_t.$$

Given the initial capital held by a household, $k_{i,0}$, and individual i's investment $\iota_{i,t}$, for $t = 0, 1, ...$, the individual stock of capital evolves according to

$$k_{i,t+1} = (1-\delta)k_{i,t} + \iota_{i,t}. \tag{5.13}$$

Given p_t, r_t, w_t, investment $\iota_{i,t}$, and labor supply $\ell_{i,t}$, for $t = 0, 1, ..$, the value of lifetime income is $\sum_{t=0}^\infty p_t(r_t k_{i,t} + w_t e_i \ell_{i,t})$ and the value of expenditures for consumption and investment is $\sum_{t=0}^\infty p_t(c_{i,t} + \iota_{i,t})$. The budget constraint requires that the value of expenditures be equal to the value of incomes, that is,

$$\sum_{t=0}^\infty p_t(c_{i,t} + \iota_{i,t}) = \sum_{t=0}^\infty p_t(r_t k_{i,t} + w_t e_i \ell_{i,t}). \tag{5.14}$$

The compact equilibrium definition reads as follows.

Definition 4 *An* ***Arrow-Debreu competitive equilibrium*** *is a set of sequences* $\left\{c_{i,t}^\star\right\}_{t=0}^\infty$, $\left\{\iota_{i,t}^\star\right\}_{t=0}^\infty$, *and* $\left\{\ell_{i,t}^\star\right\}_{t=0}^\infty$, *for each* $i \in [0,1]$, $\{L_t^\star\}_{t=0}^\infty$, $\left\{K_{t+1}^\star\right\}_{t=0}^\infty$, $\{p_t\}_{t=0}^\infty$, $\{p_t w_t\}_{t=0}^\infty$ *and* $\{p_t r_t\}_{t=0}^\infty$ *such that*

1. *for each* i, $\{c^\star_{i,t}\}_{t=0}^\infty$, $\{\iota^\star_{i,t}\}_{t=0}^\infty$, *and* $\{\ell^\star_{i,t}\}_{t=0}^\infty$ *solve*

$$\max_{\{c_t,\iota_t,\ell_t\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t \left[u(c_t) - v(\ell_t)\right] \quad \text{subject to} \quad \sum_{t=0}^\infty p_t(c_t+\iota_t) = \sum_{t=0}^\infty p_t(r_t k_t + w_t e_i \ell_t)$$

$$\text{where, } \forall t, \quad k_{t+1} = (1-\delta)k_t + \iota_t \text{ with } k_0 = k_{i,0},$$

2. $\{L^\star_t\}_{t=0}^\infty$ *and* $\{K^\star_t\}_{t=0}^\infty$, *where* $K^\star_0 = \int_0^1 k^\star_{i,0} di$, *with* $k^\star_{i,0} = k_{i,0}$, *solve*

$$\max_{\{L_t,K_t\}_{t=0}^\infty} \sum_{t=0}^\infty p_t \Big\{ A_t K_t^\alpha L_t^{1-\alpha} - r_t K_t - w_t L_t \Big\},$$

3. $\int_0^1 e_i \ell^\star_{i,t} di = L^\star_t$, $\int_0^1 k^\star_{i,t} di = K^\star_t$, *and* $\int_0^1 (c^\star_{i,t} + \iota^\star_{i,t}) di = A_t (K^\star_t)^\alpha (L^\star_t)^{1-\alpha}$ *for all* t.

Note that the $k_{i,0}$s are exogenous and that K_0 therefore is not included as an equilibrium object: we only include as equilibrium objects those that are endogenously determined. Also note, again, that since firms rent capital from consumers, their profit maximization problem is not dynamic (and we could equivalently have one firm per date). It is also possible to define an equilibrium where firms buy and own capital. A firm would then purchase a unit of the good at t and use it as an investment good and then use it, together with labor at $t+1$. That way, the firm would have a dynamic problem, making profits in period $t+1$ and on that, in equilibrium, would be just large enough to offset the costs of investment at t because of constant returns to scale.

Equilibrium characterization The optimality conditions for consumption, labor, and investment are derived by formulating the Lagrangian and taking derivatives. For consumption and labor we have, as in the previous economy,

$$\beta^t u'(c_{i,t}) = \lambda_i p_t, \tag{5.15}$$

$$v'(\ell_{i,t}) = e_i w_t u'(c_{i,t}), \tag{5.16}$$

where λ_i is the Lagrange multiplier for the (lifetime) budget constraint, equation (5.14).

Since the optimality condition for consumption is the same as in the previous economy, we can again show that all households experience the same consumption growth independently of their skills e_i and their initial wealth $k_{i,0}$, so long as $u(c)$ is a power function, which we will assume. This also implies that individual consumption is a share θ_i of aggregate consumption, that is, $c_{i,t} = \theta_i C_t$. What determines θ_i are, as before, the skills e_i and the initial wealth $k_{i,0}$, along with how prices develop over time.

The optimality condition for investment is new and it takes the form

$$p_t = p_{t+1} r_{t+1} + (1-\delta) p_{t+1}. \tag{5.17}$$

The condition has a simple interpretation. If we buy one unit of capital today, it will cost us p_t. Next period, however, we can rent it to the representative firm earning the rental rate $p_{t+1} r_{t+1}$. In addition, we still have the non-depreciated capital, $1-\delta$, which is worth

$(1-\delta)p_{t+1}$. Thus, the left-hand-side is the cost of investing and the right-hand-side is the gross return, both expressed in terms of time-0 consumption. The equality simply says that the cost of investing must be equal to its return.[5] Note that if we divide both sides of equation (5.17) by p_{t+1} we find $p_t/p_{t+1} = 1 + r_{t+1} - \delta$ so the price of consuming in date t in terms of foregone consumption in $t+1$ is given by the gross return on capital.

The firm's problem now has two optimality conditions,

$$\begin{aligned} r_t &= \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha}, \\ w_t &= (1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha}. \end{aligned}$$

The rental rate of capital and the wage rate are, respectively, the marginal productivities of capital and labor. Again, for general pairs (r_t, w_t) these two conditions will not be met at the same time (in which a firm would make infinite profits or shut down): these prices will, however, adjust so that they are, and the firm is then indifferent as to the scale of the operation. However, the capital-labor ratio is pinned down uniquely.

Let us now specialize the utility function further: $u(c) = \log c$. Then, using the first-order condition for consumption, equation (5.15), at t and $t+1$, we obtain

$$c_{i,t+1} = \beta \frac{p_t}{p_{t+1}} c_{i,t}.$$

Since $c_{i,t} = \theta_i C_t$ for all t, we obtain (equivalently, sum across all i)

$$C_{t+1} = \beta \frac{p_t}{p_{t+1}} C_t,$$

i.e., the Euler equation can also be expressed in terms of aggregate consumption. However, from the discussion at the end of Section 5.2.2 we know that aggregate labor supply will depend on the asset distribution—as given by the distribution of $k_{i,0}$s here—so let us make a simplifying assumption here:

$$k_{i,0} = e_i K_0.$$

This assumption implies if you are rich in assets, you are also rich in productivity. Here, e_i can then be interpreted also as the share of total initial capital K_0 held by agent i. We can also see that, conditional on working the same amounts, agents with different e_is will have total wealth (including the asset holding) levels proportional to e_i. To see this, use $\iota_{i,t} = k_{i,t+1} - (1-\delta)k_{i,t}$ and (5.17) for all t in the individual budget (5.14) to obtain

$$\sum_{t=0}^{\infty} p_t c_{i,t} = \sum_{t=0}^{\infty} p_t w_t e_i \ell_{i,t} + (1 - \delta + r_0) k_{i,0}.$$

Here we see that, if we set $\ell_{i,t} = \ell_t$ independent of i—a guess we will confirm to be correct momentarily—then individual i's total resources (the right-hand side) would be proportional

[5]From the consumer's perspective, one could conceive of prices and rental rates such that the stated equality is an inequality. This would either mean that a maximum does not exist or a corner solution; in equilibrium, neither are possible, so prices have to adjust to ensure equality.

to e_i, since $k_{i,0}$ also is. Since the Euler equations yield $p_{t+1}c_{i,t+1} = \beta p_t c_{i,t}$ for all t and i, we can then insert, simplify, and rewrite the budget as

$$c_{i,t} = e_i \frac{\beta^t(1-\beta)}{p_t} \left(\sum_{t=0}^{\infty} p_t w_t \ell_t + (1 - \delta + r_0) K_0 \right).$$

We see that θ_i, the individual's share of aggregate consumption, equals e_i.[6] Going back to equation (5.16), we see that the right-hand side becomes independent of i, and hence $\ell_{i,t}$ will be independent of i (recall $u'(c_{i,t}) = 1/c_{i,t}$ with log utility): it will equal L_t.

Given our assumptions, we can now think of there being a "representative consumer," even though consumers differ in endowments, since individuals' decisions all scale with e_i (this representative is, rather, the aggregate of all consumers). We have not solved the model fully yet, however: aggregates remain to be determined. We can collect them as follows:

$$\frac{C_{t+1}}{C_t} = \beta \left(1 - \delta + \alpha A_{t+1} K_{t+1}^{\alpha-1} L_{t+1}^{1-\alpha}\right), \tag{5.18}$$

where we have used the aggregate Euler equation, (5.17), and the firm's first-order condition for capital;

$$v'(L_t) = \frac{(1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha}}{C_t}, \tag{5.19}$$

where we have used the consumer's intratemporal first-order condition and the firm's first-order condition for labor; and

$$C_t = A_t K_t^{\alpha} L_t^{1-\alpha} + (1-\delta) K_t - K_{t+1} \tag{5.20}$$

from the resource constraint. Consumption can be eliminated and the system can be written as a difference equation (second-order in K, given L). It does not, in general, have a closed-form solution.[7]

How does the competitive equilibrium allocation compare to the solution to a social planner's problem of the kind we studied in Chapter 4? To explore this, let us suppose that all individuals have the same labor efficiency so $e_i = 1$ for all i. The social planner then wishes to maximize the utility of the representative household

$$\sum_{t=0}^{\infty} \beta^t \left[\log\left(C_t\right) - v\left(L_t\right)\right]$$

subject to the aggregate resource constraint (5.20). If we substitute the constraint into the objective function and take the first-order condition for the planner's problem with respect to K_{t+1} we obtain the exact same expression as (5.18). Similarly the first-order condition with respect to L_t is exactly the same as (5.19). The equations that characterize the solution to the planner's problem are therefore exactly the same as the equations that the competitive equilibrium must satisfy. As a result, the competitive equilibrium coincides with the choice of the planner. This is an important result that we will return to in Chapter 6.[8]

[6] To see this, note that $c_{i,t}$ has the structure $e_i X_t$ where X_t is independent of i. Integrating across i and using $\int e_i di = 1$ we find that $C_t \equiv \int c_{i,t} di = X_t$.

[7] $\delta = 1$ would deliver a constant saving rate equal to $\alpha\beta$, and L_t would then be given by $v'(L_t)L_t = \frac{1-\alpha}{1-\alpha\beta}$ and not depend on time.

[8] We assumed here that all the individuals have the same labor productivity to simplify the exposition,

5.3 Sequential equilibrium

In the previous sections we defined equilibria in a manner following Arrow-Debreu: all trades are decided on at time zero. In this section, instead, we assume that trades are decided sequentially over time and deliveries take place either in the same period t or in future periods. Future deliveries are especially relevant for financial contracts. For example, a debt contract is signed at time t, when borrowing occurs, but the repayment arises in one or more future periods.

Since trades arise sequentially, agents face a budget constraint in every period. This, however, does not imply that agents need to consume the whole income earned in the period. They can save by holding assets. In the rest of this section we assume that there is only one asset, denoted by a_t, that pays interest at a net rate r_t. We will define $q_t a_{t+1}$ as the amount saved at t and a_{t+1} as the amount delivered at $t+1$. This means that the real interest rate between t and $t+1$ can be defined from $1 + r_{t+1} = 1/q_t$.[9]

The period resources that the agent does not use for consumption will be used to purchase assets. If instead consumption exceeds the resources available in the period, the agent will borrow. Borrowing means that the value of a_t is negative, in which case the agent will pay an interest (the interests on borrowing and saving are the same). This will become clear in the applications.

5.3.1 The endowment economy

Returning to our endowment economy from Section 5.2.1, let us now assume that agents trade in every period. Agents trade an asset that pays the interest rate r_t: agent i enters period t with $a_{i,t}$ units of the asset and receives the endowment $y_{i,t}$. Therefore, the total resources available in the period are $a_{i,t} + y_{i,t}$. These resources are then used in part for consumption, $c_{i,t}$, and in part to purchase new units of the asset, $q_t a_{i,t+1}$. The budget constraint in period t is

$$c_{i,t} + q_t a_{i,t+1} = y_{i,t} + a_{i,t}.$$

The agent maximizes lifetime utility (5.1) subject to the sequence of budget constraints, one for every period, and the no Ponzi game (nPg) condition introduced in Section 4.3.1. The initial asset holdings, which are exogenous, sum to zero: $\int_0^1 a_{i,0} di = 0$. We have the following.

Definition 5 *A* ***sequential competitive equilibrium*** *is a set of sequences* $\left\{c^\star_{i,t}\right\}_{t=0}^{\infty}$ *and* $\left\{a^\star_{i,t+1}\right\}_{t=0}^{\infty}$*, for each* $i \in [0,1]$*, and* $\{q_t\}_{t=0}^{\infty}$ *such that*

but this assumption is not crucial. With heterogeneous skill levels we could show that the competitive equilibrium coincides with the solution to a different social planner's problem in which the social planner maximizes a weighted average of the utilities of the individuals with weights given by e_i.

[9] This convention is common; in an endowment economy, if alternatively a_{t+1} is saving in consumption units at t and $(1 + r_{t+1})a_{t+1}$ is the total return from this saving at $t+1$, then r_0 cannot be determined in equilibrium (or, equivalently, it can be set to any value). However, q_0 is determined.

1. *for each i, $\{c^\star_{i,t}\}_{t=0}^\infty$ and $\{a^\star_{i,t+1}\}_{t=0}^\infty$ solve*

$$\begin{aligned}
\max_{\{c_t,a_{t+1}\}_{t=0}^\infty} \quad & \sum_{t=0}^{\infty} \beta^t u(c_t) \\
& \textit{subject to} \\
& c_t + q_t a_{t+1} = a_t + y_{i,t} \quad \forall\ t,\ \textit{with } a_0 = a_{i,0}, \\
& \lim_{t\to\infty} \left(\prod_{s=0}^{t} q_s \right) a_{t+1} \geq 0. \qquad \textit{(nPg condition)}
\end{aligned}$$

2. *for all t, $\int_0^1 c^\star_{i,t} di = \int_0^1 y_{i,t} di$ and $\int_0^1 a^\star_{i,t+1} di = 0$.*

The last two conditions are the market clearing conditions in goods and financial markets. The first says that the aggregate consumption must be equal to the aggregate quantity of goods available in every period. The second says that aggregate asset holdings must be zero in every period. In this economy, agents borrow and lend to one another. Each transaction therefore represents an increase in one person's assets and an offsetting decrease in another's. When we sum across all the agents, these trades net out to zero.

The equilibrium definition imposes two market clearing conditions. Because Walras' Law applies, only one of the two market-clearing conditions needs to be imposed. If we integrate the household budget constraints across i and impose asset market clearing we arrive at the goods market clearing condition. Hence goods market clearing is insured by asset market clearing. It is also the case that goods market clearing implies the asset market clears.[10]

The optimality conditions can be derived by writing the Lagrangian and taking first-order conditions

$$\beta^t u'(c_{i,t}) = \lambda_{i,t} \tag{5.21}$$

and

$$q_t \lambda_{i,t} = \lambda_{i,t+1}, \tag{5.22}$$

where $\lambda_{i,t}$ is the Lagrange multiplier associated with the budget constraint. Differently from the first-order conditions in the setup under date-zero trading, the multiplier depends on time t. This is because we replaced the lifetime budget constraint (which is one constraint) with the sequence of period budget constraints. Therefore, we have one multiplier associated with each period budget constraint.

We now show that, despite this difference, we obtain the same optimality conditions. Using equation (5.21) at time t and $t+1$ to eliminate the multipliers in equation (5.22) we obtain

$$q_t = \beta \frac{u'(c_{i,t+1})}{u'(c_{i,t})}.$$

[10]If we integrate the budget constraints at time 0 and impose $\int_0^1 a_{i,0} di = 0$ and goods market clearing we find $\int_0^1 a_{i,1} di = 0$. We can then proceed by induction to show that goods market clearing at date 1 implies asset market clearing at date 1 and so on.

This condition must be satisfied at any time t. If we use this condition from time 0 through time $t-1$, we obtain

$$q_0 \times q_1 \times \cdots \times q_{t-1} = \beta \frac{u'(c_{i,1})}{u'(c_{i,0})} \times .. \times \beta \frac{u'(c_{i,t})}{u'(c_{i,t-1})},$$

which can be rewritten more compactly as

$$\prod_{s=0}^{t-1} q_s = \beta^t \frac{u'(c_{i,t})}{u'(c_{i,0})}.$$

The left-hand-side term corresponds to p_t in the Arrow-Debreu equilibrium. Therefore, we obtained the optimality condition (5.3) we derived in the date-zero trade equilibrium. This illustrates that the Arrow-Debreu price p_t is the present value at time zero of one unit of consumption at time t, discounted by the sequence of interest rates up to time t. It also shows that $p_{t-1}/p_t = 1/q_{t-1} = 1 + r_t$, that is, the gross interest rate between $t-1$ and t is the ratio between the corresponding Arrow-Debreu prices: the relative price of consumption goods at $t-1$ in terms of goods at t.

The sequential equilibrium delivers the same equations determining quantities and, once translated back into Arrow-Debreu terms, the same prices as well. The literature uses both, guided by what is convenient in different contexts.

It is instructive to connect the budget constraints in the two setups: at first appearance they may not look equivalent, but they are. To see this, we start with the budget constraints at $t = 0$ and $t = 1$:

$$\begin{aligned} c_{i,0} + q_0 a_{i,1} &= a_{i,0} + y_{i,0}, \\ c_{i,1} + q_1 a_{i,2} &= a_{i,1} + y_{i,1}. \end{aligned}$$

Using the first equation to eliminate $a_{i,1}$ in the second equation (or viceversa) we obtain

$$c_{i,0} + q_0 c_{i,1} + q_1 q_0 a_{i,2} = a_{i,0} + y_{i,0} + q_0 y_{i,1}.$$

We use next the budget constraint at $t = 2$ to eliminate $a_{i,2}$, then the budget constraint at $t = 3$ to eliminate $a_{i,3}$, and so on. After T substitutions we obtain

$$\sum_{t=0}^{T} \left(\prod_{s=0}^{t-1} q_s \right) c_{i,t} + \left(\prod_{s=0}^{T-1} q_s \right) a_{i,T+1} = a_{i,0} + \sum_{t=0}^{T} \left(\prod_{s=0}^{t-1} q_s \right) y_{i,t}.$$

We have already shown that $\prod_{s=0}^{t-1} q_s = p_t$. Furthermore, as T converges to infinity, the second term on the left-hand side of the equation converges to something non-negative. Therefore, taking the limit $T \to \infty$ we obtain the lifetime budget constraint

$$\sum_{t=0}^{\infty} p_t c_{i,t} \leq a_{i,0} + \sum_{t=0}^{\infty} p_t y_{i,t}.$$

The budget constraint will be chosen to hold with equality given a strictly increasing utility function; that is, the limit of $\left(\prod_{s=0}^{T-1} q_s \right) a_{i,T+1}$ as t approaches infinity will be zero (a strictly positive amount would violate the TVC discussed in Chapter 4).

It is straightforward to apply a sequential equilibrium concept to other economies; to save space, we will only look at the neoclassical growth model, this time without valued leisure.

5.3.2 The neoclassical growth economy

We can similarly define an equilibrium of the neoclassical growth model in which agents trade period by period. We simply state the compact equilibrium definition here and leave it up to the reader to verify that it is equivalent to that described as an Arrow-Debreu equilibrium: that the allocations coincide and that prices, appropriately defined in comparable terms, do too. We use the case where labor is set exogenously to e_i for agent i, with $\int_0^1 e_i di = 1$ so that aggregate labor supply equals 1.

Definition 6 *A* ***sequential competitive equilibrium*** *is a set of sequences* $\left\{c^\star_{i,t}\right\}_{t=0}^\infty$ *and* $\left\{k^\star_{i,t+1}\right\}_{t=0}^\infty$*, for each* $i \in [0,1]$*, and* $\{r_t\}_{t=0}^\infty$ *and* $\{w_t\}_{t=0}^\infty$ *such that*

1. *for each* i*,* $\left\{c^\star_{i,t}\right\}_{t=0}^\infty$ *and* $\left\{k^\star_{i,t+1}\right\}_{t=0}^\infty$ *solve*

$$\begin{aligned}
\max_{\{c_t,k_{t+1}\}_{t=0}^\infty} \quad & \sum_{t=0}^{\infty} \beta^t u(c_t) \\
& \text{subject to} \\
& c_t + k_{t+1} = (1-\delta+r_t)k_t + w_t e_i \quad \forall\, t, \;\; \text{with } k_0 = k_{i,0}, \\
& \lim_{t\to\infty} \frac{1}{\prod_{s=0}^{t}(1-\delta+r_{s+1})} k_{t+1} \geq 0 \qquad \text{(nPg condition)};
\end{aligned}$$

2. *for each* t*,* $(K_t, 1)$ *solves* $\max_{K,L} A_t K^\alpha L^{1-\alpha} - r_t K - w_t L$*, where* $K_t = \int_0^1 k^\star_{i,t} di$ *(and* $k^\star_{i,0} = k_{i,0}$*); and*

3. *for each* t*,*
$$C_t + K_{t+1} = A_t K_t^\alpha + (1-\delta)K_t,$$
where $C_t = \int_0^1 c^\star_{i,t} di$*.*

Let us make two remarks. First, the use of Walras's Law here makes the last requirement redundant: add consumers' budgets at each point in time and use the firm's problem, which has constant returns to scale, to see this.[11] Second, we formulate the firm's problem as static here; it is simpler. In it, note that whereas K and L are mere choice variables and need not have time subscripts, the solution does need to be dated.

5.4 Recursive equilibrium

Just like it is possible to study dynamic optimization problems with recursive methods—dynamic programming—it is also possible to define equilibria that way. We saw in the context of optimization that recursive methods make the object of study different: we look for functions, not sequences. For a maximization problem, having solved a dynamic program means that we have a function that we can apply to any value of its argument: the state variable (which can be a vector). A recursive equilibrium will also focus on functions.

[11] Walras's Law was applicable in the Arrow-Debreu formulation too, but only once, not at each period t.

To move slowly toward a general definition, let us first study how recursive methods can be used to define steady-state equilibria, i.e., equilibria where (aggregate) variables are constant over time. In the case of equilibria defined as sequences, we did not separately define steady-state equilibria: steady-state equilibria simply satisfy the definition of equilibria and have the additional property that all variables (at least aggregates) are constant over time. For the endowment economy, the case where all individuals' endowments are constant over time, all equilibria are steady-state equilibria. For the neoclassical economy, the economy's aggregate capital stock needs to have a certain starting value; otherwise, the equilibrium will not be in steady state from time zero. Steady-state equilibria defined using recursive methods are also just a subset of all equilibria. However, as we shall see, it is useful to define steady-state equilibria separately, as a first step.

5.4.1 Steady state

A steady-state equilibrium is one where the aggregate economy is at a rest point. This means, in particular, that prices are constant. Before proceeding, note that the present discussion also applies to the case where there is exact balanced growth, in which case the economy can be transformed into a stationary one whose steady state is then studied (in this case, wages will be growing along the balanced path but will be constant in the transformed economy).

Conceptually, now, for the definition of a steady-state equilibrium we will need to specify the constant prices and aggregates and to provide functions making clear that, at steady state, agents optimize: they consider behavior that is non-constant over time but choose to remain constant.

As before, we will go through definitions for different economies, beginning with a case without production.

The endowment economy

Consider, again, the economy with a continuum of consumers indexed by i, each now with and endowment that is constant over time, y_i. An equilibrium is defined as follows.

Definition 7 *A* ***recursive steady-state competitive equilibrium*** *is a q and a set of functions, $V_i^\star(a)$ and $g_i^\star(a)$, and asset values $a_i^\star$, for each $i \in [0,1]$ such that*

1. *for each i, $V_i^\star(a)$ solves*

$$V_i(a) = \max_{a'} u(a + y_i - qa') + \beta V_i(a') \quad \forall a,$$

   with $g_i^\star(a)$ attaining the maximum on the right-hand side for all a;

2. *for each i, $g_i^\star(a_i^\star) = a_i^\star$; and*

3. $\int_0^1 g_i^\star(a_i^\star)di = 0.$

Here, the $V^\star$s and $g^\star$s capture individual optimization, given prices (a single q in this case). The second condition expresses stationarity of individual asset holdings; at the same time, it does put a sharp restriction on the function. We are allowing different individuals to start with—and maintain—different holdings of assets. Market clearing is captured in the third condition. Our equilibrium definition also specifies a distribution of asset holdings, which is required to be constant over time. A special case is that of a representative agent, where the second condition would read $g^\star(0) = 0$ and the third condition would be superfluous.

Equilibrium characterization The functional Euler equation, which we arrive at after taking the first-order condition in the dynamic program and then using the envelope theorem, reads, for all i,

$$qu'\left(a + y_i - qg_i(a)\right) = \beta u'\left(g_i(a) + y_i - qg_i(g_i(a))\right).$$

Using stationarity at $a = a_i$, we obtain

$$qu'\left(a_i(1-q) + y_i\right) = \beta u'\left(a_i(1-q) + y_i\right).$$

Hence, $q = \beta$. We can now solve for g_i from the functional Euler equation: it implies $a - qg_i(a) = g_i(a) - qg_i(g_i(a))$ and it is easy to see here that $g_i^\star(a) = a$ solves this functional equation for all i. No matter what asset level an agent has, they choose to keep it and just consume its accrued interest. This is the permanent-income result we have seen before, now in recursive form.

Given the simple form of the policy function, we immediately obtain the implied $V_i^\star$s. They satisfy $V_i^\star(a) = u(a(1-q) + y_i)/(1-\beta)$.

Having found all the equilibrium objects, we note, first, that the asset distribution can be chosen freely subject to market clearing and that all consumers have non-negative consumption. That is, this model has "no predictions" for long-run wealth distributions: any relative distribution of wealth is sustained over time. Since real-world wealth distribution have certain distinct features in common—regarding their overall shapes, across time and countries—this model therefore is not satisfactory. Richer models of wealth distribution that relate to data are therefore developed and discussed in Chapter 21. Second, we see that the shape of the utility function, u, does not play a role in any of the characterizations of the steady state. This is not true if there is balanced growth (in endowments); then, u needs to have the usual balanced-growth shape.

The neoclassical growth economy

Turning to the neoclassical growth model with optimal saving, let us proceed immediately to the definition of steady-state equilibrium, of course with the assumption that TFP, A_t, is constant over time.[12] We will again consider a continuum of consumers. For simplicity, we will assume that labor supply is exogenous and equal to e_i for agent i, with $\int_0^1 e_i di = 1$.

[12] Alternatively, it is growing at a constant rate, in which case the analysis requires a transformation of variables.

Definition 8 *A* ***recursive steady-state competitive equilibrium*** *consists of scalars r and w and a set of functions, $V_i^\star(k)$ and $g_i^\star(k)$, and capital holdings $k_i^\star$, for each $i \in [0,1]$ such that*

1. *for each i, $V_i^\star(k)$ solves*
$$V_i(k) = \max_{k'} u((1-\delta+r)k + we_i - k') + \beta V_i(k') \quad \forall k,$$
with $g_i^\star(k)$ attaining the maximum on the right-hand side for all k;

2. *$(K^\star, 1)$ solves $\max_{K,L} F(K,L) - rK - wL$, where $\int_0^1 k_i^\star di = K^\star$; and*

3. *for each i, $g_i^\star(k_i^\star) = k_i^\star$.*

We see a close similarity between the agent's problem here and that in the endowment economy: there exogenous labor income and a constant interest rate. There are two prices, r and w, but as we shall see they are closely related. Finally, we see that the market-clearing condition is different: assets sum up to the aggregate capital stock.

Equilibrium characterization For reasons identical to those used in the case of the endowment economy, we obtain $\beta = 1 - \delta + r$. This equation determines r. Then we know from firm maximization that $r = F_1(K^\star, 1)$, where $F_i(\cdot,\cdot)$ represents the partial derivative with respect to ith argument. This equation determines $K^\star$. We also know that $w = F_2(K^\star, 1)$, which gives us w. The distribution of capital is indeterminate, subject to adding up to $K^\star$. The policy and value functions are given by $g_i^\star(k) = k$ and $V_i^\star(k) = u(k(r-\delta) + we_i)/(1-\beta)$, respectively, for all k and i.

5.4.2 Dynamics

Turning to a full equilibrium, we now need to find a way to express, using functions, how prices and aggregates move over time. We will consider the neoclassical growth model only.[13] We will assume that TFP is constant, so as to isolate how these price and quantity movements are endogenous. We will, again, consider a continuum of households, but now assume as a benchmark that their asset holdings and labor productivities are the same. This means that the consumer we look at can be thought of as a representative agent from the outset, i.e., one among a [0,1] continuum of identical agents. We will revisit the question of actual heterogeneity at the end. We also begin with the assumption that labor supply is exogenous and equal to 1.

Recall that recursive methods involve expressing outcomes as functions of *state variables*. A state variable has to be both relevant and predetermined. So, in our economy, what determines prices and aggregates over time? It is instructive to consider period 0: what determines prices at that point in time? In the neoclassical model r_0 and w_0 are determined by the capital/labor ratio, through the marginal products of capital and labor. Hence, at the

[13] Endowment economies can be considered as well but do not involve the same core issues as the neoclassical model allows us to illustrate.

very least these two prices will depend on the period-0 capital stock in the economy: besides being relevant, the aggregate capital stock is also predetermined. Is there another variable that qualifies as a state, determining prices? No. Hence, we conclude that two functions $r(K)$ and $w(K)$ need to be part of our equilibrium.[14]

Agents will take as given $r(K)$ and $w(K)$ when making their decisions. But, given that prices will move over time, how do they know what prices will prevail in the future? For this they need the equivalent of the planner's decision rule for capital: in the context of a recursive equilibrium, we will label it a law of motion, $K' = G(K)$. Thus, agents take G, r, and w as given functions and then solve their dynamic programming problems. What will these problems look like? As in the case of the steady-state equilibrium we need to allow the agent's choice of capital to deviate from that in equilibrium. Therefore the dynamic program must read

$$V(k,K) = \max_{k'} u((1-\delta+r(K))k + w(K) - k') + \beta V(k', G(K)) \quad \forall (k,K), \tag{5.23}$$

where k is an individual agent's capital and K is the aggregate capital stock, which they take as given.

We now define equilibrium:

Definition 9 *A* ***recursive competitive equilibrium*** *consists of functions $r(K)$, $w(K)$, $G^\star(K)$, $V^\star(k,K)$, and $g^\star(k,K)$ such that*

1. *$V^\star(k,K)$ solves (5.23), for $G = G^\star$ and $g^\star(k,K)$ attains the maximum in this problem;*
2. *for all K, $r(K) = F_1(K,1)$ and $w(K) = F_2(K,1)$; and*
3. *$G^\star(K) = g^\star(K,K)$ for all K.*

The second condition does not state profit maximization of the firm explicitly but just uses the first-order conditions.[15] The third condition, labeled *consistency*, works as a market clearing condition: it requires that the evolution of the aggregate capital stock is consistent with the choices of individuals when they each hold the same level of capital.

Is the resource constraint met in our definition? It is. The representative consumer will, in a recursive competitive equilibrium, consume $K(1-\delta+r(K)) + w(K) - G^\star(K)$, which, from the firm's problem and F being CRS, equals $F(K,1) - K' + K(1-\delta)$, i.e., output minus investment.

Equilibrium characterization Let us also derive the functional Euler equation of the agent. Taking first-order conditions and applying the envelope theorem (for a marginal change in k), we obtain (dropping *s for convenience) that,for all (k,K),

$$\begin{aligned} &u'\left((1-\delta+r(K))k + w(K) - g(k,K)\right) = \\ &\qquad \beta u'\left(g(k,K)(1-\delta+r(G(K))) + w(G(K)) - g(g(k,K),G(K))\right)\left[1-\delta+r(G(K))\right]. \end{aligned}$$

[14] As an alternative to looking at time 0 to gain intuition, consider the planning problem of the economy under study: the state variable relevant to the planner will then be a state in the recursive equilibrium.

[15] Alternatively, state that $r(K)$ and $w(K)$ are such that $(K,1)$ solves $\max_{k,\ell} F(k,\ell) - r(K)k - w(K)\ell$ for all K.

Given r, w, and G, this functional equation determines $g(k, K)$: the behavior of individual saving when the individual has k, possibly different from aggregate capital, K. Can g be solved for explicitly? In general, no. However, when u is a power function, again in line with our general balanced-growth requirements, it is possible to show that it takes the form $g(k, K) = \mu(K) + \lambda(K)k$.[16] Individual saving is *linear* in the own holdings of capital: the marginal propensity to save is $\lambda(K)$, i.e., independent of the level of k (it only depends on the aggregate capital stock). This also means that if we were to consider several consumers and distribute capital among them, how we distribute it would not matter for aggregate saving. In other words, we have *aggregation* if u is a power function. Thus, it is not restrictive to consider a representative agent. This result holds also if different agents have different endowments of labor: then $g_i(k, K) = \mu_i(K) + \lambda(K)k$, i.e., the intercept will vary across individuals but the slope will not. To conclude, if an economy with a power utility function has a nontrivial distribution of capital among agents, the aggregate law of motion will only depend on aggregate capital. That is, K is still the aggregate state variable: the distribution of capital, though predetermined, is not relevant for understanding how prices are determined.

Turning to how one solves for aggregates, we can also evaluate the above Euler equation at $k = K$ and use the fact that the resource constraint will hold, along with the expression for the rental rate function, to obtain

$$\begin{aligned}&u'\left(K(1-\delta)K + F(K,1) - G(K)\right) = \\ &\qquad \beta u'\left(G(K)(1-\delta) + F(G(K),1) - G(G(K))\right)\left[1 - \delta + F_1(G(K),1)\right].\end{aligned}$$

This functional equation solves for the evolution of the capital stock. We note that it coincides with the functional equation of the planner's problem for the same economy: equation (4.26) of our optimization chapter. As in that case, this equation has to be solved numerically unless u is logarithmic, F is Cobb-Douglas, and $\delta = 1$.

Before concluding, let us consider the economy with valued leisure: agents have utility functions $u(c) - v(\ell)$. Now aggregate labor supply is endogenous. It is not predetermined in a given period, so it is not a state variable (neither for the individual nor for the aggregate). Thus, our state is still (k, K). What is needed now, however, is a function $L = H(K)$ specifying how aggregate labor supply depends on the aggregate state. Similarly, on the individual level, we need $\ell = h(k, K)$ to denote the policy function for the choice of labor. The equilibrium, again for the representative-agent case, becomes

Definition 10 *A* ***recursive competitive equilibrium*** *for the economy with valued leisure consists of functions $r(K)$, $w(K)$, $G^\star(K)$, $H^\star(K)$, $V^\star(k, K)$, $g^\star(k, K)$, and $h^\star(k, K)$ such that*

1. *$V^\star(k, K)$ solves*

$$V(k, K) = \max_{k', \ell} u((1 - \delta + r(K))k + w(K)\ell - k') - v(\ell) + \beta V(k', G^\star(K)) \quad \forall (k, K).$$

and $k' = g^\star(k, K)$ and $\ell = h^\star(k, K)$ attain the maximum in this problem;

[16] See Appendix 5.A for a proof and a more extensive discussion of conditions under which the solution has this form.

2. *for all* K, $r(K) = F_1(K, H^\star(K))$ *and* $w(K) = F_2(K, H^\star(K))$*; and*

3. $G^\star(K) = g^\star(K, K)$ *and* $H^\star(K) = h^\star(K, K)$ *for all* K.

This is a straightforward extension of the definition without valued leisure. Notice that the agent does not use $H^\star$ in the maximization problem; $G^\star$ suffices, because r and w now capture how the labor input changes with K.

It is straightforward, but somewhat tedious, to derive the functional Euler equations; there will now be an intertemporal condition too determining labor supply. Again, evaluated at $k = K$, one finds that the conditions are identical to those of the corresponding planner's problem.

Will this economy deliver aggregation too, once u is of the power form? As already alluded to above in Section 5.2.3, the answer is no, unless labor productivities differ across agents too and the ratio of the initial capital holding to labor productivity is the same across all agents.[17] If not, the aggregate state variable necessarily becomes the vector of capital holdings of all agents, not just the sum of these holdings.

5.5 Overlapping generations

We now consider overlapping generations models. The defining feature of these models is that agents live for a finite length of time but the economy continues after their death with new generations of agents. Historically, overlapping-generations (OG) models have played an important role in macroeconomics. They were first introduced by Allais (1947) and later used for a large variety of purposes: for understanding why fiat money has value and the potential need for a social security system (Samuelson, 1958) and for understanding government debt (see Diamond, 1965). Interestingly, although most of his later work used dynastic settings, Lucas's path-breaking (see Lucas, 1972) paper on the Phillips curve is using an OG model. We will revisit these applications later in the text. In this chapter, we will merely focus on some features that make OG market economies different than those studied above.

We will restrict attention to the simplest version of an OG setting: one where, each period, a cohort of people are born and then live for two periods only. That way, cohorts overlap, but only for one period. This is the so-called the two-period life OG setting. Clearly, the two-period life case is limited in its applicability: for example, it cannot be used for quantitative studies of the business cycle, as business cycles occur rather frequently (a time period in the OG model should perhaps be thought of as 25 or 30 years). However, it is of course possible to construct OG models where people live for an arbitrary (finite) number of periods and many of the special properties of such models are inherited from the two-period life case.

In our benchmark, we will assume—as is the case in the most commonly used OG model—that people, though they have children, do not give bequests or any other gifts to them. Relatedly, they express no altruism toward their offspring and maximize their own utility only, defined over consumption (and possibly leisure) in their two periods of life. The maximization problems are, then, conceptually simple; the finite-horizon settings studied in Section 4.2, in particular that in 4.2.1, can be immediately applied. The maximization problems,

[17] In this case, this ratio will also stay the same over time.

moreover, are often called *life-cycle models*, emphasizing that individuals go through different phases in life and, in particular, that young and old individuals have different time horizons. Here, we will limit heterogeneity to age and not consider further differences between people. Therefore, within each cohort, all agents are identical.

Defining equilibria for OG models is straightforward, too. As for the dynastic setting, there are three methods—sequence-based (AD and sequential-trade) and recursive definitions of equilibria—but here, for convenience only, we will focus exclusively on the sequential-trade setting. We will see that in OG models, not only are the maximization problems simpler, but computing equilibria is also more straightforward. However, what we will find is that equilibria for OG models can (but do not necessarily) have peculiar features. For example, equilibria may not be Pareto optimal. In addition, there may be more than one equilibrium. In this chapter, we will mainly set things up; welfare properties will then be studied carefully in the next chapter and applications will be discussed in later chapters.

As before, we begin with the endowment case. We will then turn to the neoclassical growth model. Lastly, we will discuss introducing altruism and bequests. Throughout, we abstract from population growth.

5.5.1 The endowment economy

Let us assume that the cohort born at t has utility given by

$$u_t(c_y,\, c_o) = u(c_y) + \beta u(c_o),$$

where we evaluate at two arbitrary consumption levels (c_y, c_o) when young and old, respectively. The preferences of generation $t = -1$, who are old as time begins, are similarly represented by $u_{-1}(c) = u(c)$.

We consider endowment sequences given by $(\omega_{y,t}, \omega_{o,t+1})$ for cohort t: for all t, where t is the time period, $\omega_{y,t}$ is the endowment of the young and $\omega_{o,t}$ the endowment of the old (who are of cohort $t-1$) in that time period.

A sequential equilibrium is defined in much the same way as in our endowment economy with dynastic agents.

Definition 11 *A* ***sequential competitive equilibrium*** *is a set of sequences* $\left\{c^\star_{i,t}\right\}_{t=0}^{\infty}$, *for each* $i \in \{y, o\}$, $\left\{a^\star_{t+1}\right\}_{t=0}^{\infty}$, *and* $\{q_t\}_{t=0}^{\infty}$ *such that*

1. *for each* $t > 0$, $\left(c^\star_{y,t}, c^\star_{o,t+1}, a^\star_{t+1}\right)$ *solves*

$$\max_{c_y, c_o, a'} u(c_y) + \beta u(c_o) \quad \textit{subject to} \quad c_y + q_t a' = \omega_{y,t} \textit{ and } c_o = \omega_{o,t+1} + a'$$

   and $c_{o,0} = \omega_{o,0}$.

2. *for all* $t \geq 0$, $c^\star_{y,t} + c^\star_{o,t} = \omega_{y,t} + \omega_{o,t}$.

The last requirement—goods market clearing—can equivalently, using consumers' budgets, be written $a^\star_{t+1} = 0$ for all t. To see this, begin in period 0 and roll forward: resource feasibility at time 0 means, when summing cohort 0's budget and cohort 1's first-period budget, that $a_1^\star$ must equal 0. Then the same procedure for next period delivers $a_2^\star = 0$, and so on. Intuitively, this is obvious: any given cohort fundamentally only have endowment income, and saving (or borrowing) when young must mean that another cohort is on the other side of that transaction; but it cannot be the old who is now alive during their last period, and it cannot be the young of next cohort, since they have not been born yet.

Equilibrium characterization Having established $a_{t+1}^\star = 0$, a result that is true also in the dynastic model if all agents have the same endowments, we conclude that $(c_{y,t}^\star, c_{o,t}^\star) = (\omega_{y,t}, \omega_{o,t})$: autarky. These results are in line with what we found in the dynastic model. Also, as before, solving for the price only involves evaluating the consumer's first-order (Euler) condition, which—evaluated at autarky—reads

$$q_t u'(\omega_{y,t}) = \beta u'(\omega_{o,t+1})$$

at time t. If we impose $u(c) = (c^{1-\sigma} - 1)/(1-\sigma)$, so that (in a slightly more general model) we obtain results consistent with balanced growth, we conclude that

$$q_t = \beta \left(\frac{\omega_{y,t}}{\omega_{o,t+1}} \right)^\sigma . \tag{5.24}$$

If the endowments are stationary, so that $\omega_{y,t}$ and $\omega_{o,t}$ do not depend on time, we obtain that q_t is constant and equal to $\beta(\omega_y/\omega_o)^\sigma$. Thus, if the endowment when young does not equal the endowment when old, the interest rate will not just reflect the utility discount factor β but also the shape of *life-cycle income*. In particular, if $\omega_y > \omega_o$, which makes sense if we identify y with "working" and o with "being retired," then we can even obtain $q > 1$, i.e., negative real interest rates, in this model. This is not possible to obtain in the dynastic endowment model (unless aggregate endowments fall over time at a constant, and high enough, rate).

The intuition for the possibility of negative interest rates in the OG model is that when life-cycle endowments decline over time (at a high enough rate), consumption is marginally more valuable in the future than now in the absence of being able to smooth income over time. And in the two-period life OG model no such smoothing is feasible. In an OG model where people live for three periods there could be borrowing/lending between the young and the middle-aged. It would be reasonable to think that the life-cycle endowment pattern then has $\omega_y < \omega_m$ and $\omega_m > \omega_o$, where ω_m is the endowment of the middle-aged. The young then could borrow from the currently middle-aged, who want to lend. Thus, some consumption smoothing would still occur, and the discount factor would again play a more prominent role; however, full smoothing would not necessarily emerge. As a result, the OG model continues to yield predictions that differ from those of the dynastic model. Overall endowment growth would also affect the result in the direction of producing higher real interest rates.

Finally, note that it would be beneficial to transfer resources from the young to the old, in the case where $\omega_y > \omega_o$. If *all* young transfer to the current old then it is even possible for *all* generations to benefit. This kind of transfer can be thought of as a government-run pay-as-you-go pension scheme; in fact, this is a key early use of the OG model. Thus, we have an indication that the market is not efficient here. We will discuss this issue at much greater length in our welfare chapter: Chapter 6.

5.5.2 The neoclassical growth economy

In the two-period life OG model, the introduction of capital allows consumers to engage in life-cycle saving. They are endowed with labor income both when young and old and, as before, have no initial assets. So, if their income when young exceeds their income when old, they would buy capital when young and rent it out, as old agents, to firms. At time 0, the capital stock is owned by the initial old.

For concreteness, we let the labor productivity (or alternatively time endowment) of the young and old at all times be e_y and e_o, respectively, with $e_y + e_o = 1$. We also restrict attention to a stationary production function denoted by $F(k, \ell)$, with F_1 and F_2 representing the marginal products of capital and labor, respectively. Growth in productivity is straightforward to introduce and it is not essential for the key points here. We have the following (where we use a representative agent within each cohort to economize on notation).

Definition 12 *A* ***sequential competitive equilibrium*** *is a set of sequences* $\{c^\star_{i,t}\}_{t=0}^{\infty}$*, for each* $i \in \{y, o\}$*,* $\{k^\star_{t+1}\}_{t=0}^{\infty}$*,* $\{r_t\}_{t=0}^{\infty}$*, and* $\{w_t\}_{t=0}^{\infty}$ *such that*

1. *for each* $t > 0$*,* $\left(c^\star_{y,t}, c^\star_{o,t+1}, k^\star_{t+1}\right)$ *solves*

$$\max_{c_y, c_o, k'} u(c_y) + \beta u(c_o) \quad \text{subject to} \quad c_y + k' = w_t e_y \text{ and } c_o = w_{t+1} e_o + (1 - \delta + r_{t+1})k'$$

and $c_{o,0} = w_0 e_o + (1 - \delta + r_0)k_0$*;*

2. *for all* t*,* $r_t = F_1(k^\star_t, 1)$ *and* $w_t = F_2(k^\star_t, 1)$*, with* $k^\star_0 \equiv k_0$*; and*

3. *for all* $t \geq 0$*,* $c^\star_{y,t} + c^\star_{o,t} + k^\star_{t+1} = F(k^\star_t, 1) + (1 - \delta)k^\star_t$*.*

This definition is similar to that in a dynastic economy, except that there are two types of agents, only one of which saves in capital at any point in time.

Equilibrium characterization The focus is on the individual problem. The Euler equation is entirely standard-looking, so let us proceed immediately to evaluate it after expressing prices as a function of capital stocks:

$$\begin{aligned} &u'\left(e_y F_2(k^\star_t, 1) - k^\star_{t+1}\right) = \\ &\quad \beta u'\left(e_o F_2(k^\star_{t+1}, 1) + (1 - \delta + F_1(k^\star_{t+1}, 1))k^\star_{t+1}\right)\left(1 - \delta + F_1(k^\star_{t+1}, 1)\right). \end{aligned}$$

Conditional on a value for $k^\star_t$, this equation solves for $k^\star_{t+1}$. Therefore, we have a very different dynamic system than in the dynastic model, where we always obtained a second-order difference equation. This means that we can "solve forward": start with k_0, solve for $k^\star_1$, and so on.

Second, what does a steady state look like? Letting $\bar{k}$ denote a steady state, we obtain

$$u'\left(e_y F_2(\bar{k}, 1) - \bar{k}\right) = \beta u'\left(e_o F_2(\bar{k}, 1) + (1 - \delta + F_1(\bar{k}, 1))\bar{k}\right)\left(1 - \delta + F_1(\bar{k}, 1)\right).$$

That is, we do not, in general, obtain $\beta\left(1 - \delta + F_1(\bar{k}, 1)\right) = 1$, since consumption may not end up being fully smoothed. Again, we see that OG models have qualitatively different implications for long-run interest rates: they can, depending on parameter values (intuitively, depending on the demand for saving given the life-cycle structure, and depending on firm's demand for capital) be either higher or lower than $1/\beta$ (in gross terms). We invite the reader to fully solve the model where u is logarithmic, $e_y = 1$ and $e_o = 0$, F is Cobb-Douglas, and $\delta = 1$ and verify that (i) capital's dynamics will be log-linear and converge monotonically to a steady state; and (ii) the gross steady-state interest rate will be $\alpha(1+\beta)/(1-\alpha)$, a number which is less than one if α is low enough.

Finally, we also note that although the solution for the model's dynamics involves only a first-order difference equation, it is not immediately obvious that, for each k_t, there is a unique value k_{t+1} solving the Euler equation. In our dynastic model, we obtain a unique equilibrium so long as it can be obtained as a solution to a planner's problem, which we know to be unique under standard assumptions. Here, there is no immediate connection to a planner's problem.[18]

5.5.3 Some model comparisons

We have studied the OG model in its two-period life version. When consumers live for more than two periods, the model in some ways looks more like a dynastic model; as the time horizon gets longer, the life-cycle patterns can become less pronounced. Also, solving for equilibria in OG models with more than two-period lives involves difference equations that are of higher order than one. Still, they are conceptually different. The determination of the long-run real interest rate, for example, remains more complex than in the dynastic model, where the equilibrium interest rate depends on very few parameters: without aggregate growth, the gross real interest rate is $1/\beta$, and with net growth at rate γ, it is $(1+\gamma)^\sigma/\beta$.

Random deaths There are other variants of the OG model. One is the *perpetual-youth* (or, perhaps, *sudden-death*) model. There, all individuals face a constant probability of death between t and $t+1$. Thus, a "lucky" individual can live a very long time, potentially forever. As a consequence, anybody alive at t has the same expected remaining lifetime. An individual who dies will be replaced by a new-born individual, so as to avoid a shrinking population, but again not as part of a dynasty: this model shares with the basic OG setting that no individual cares about children. When an individual dies with unspent assets—which is typical—then these assets can be either seized by the government or given as "random bequests" to the surviving population. A third alternative—all have been used in the literature—is that agents can write a form of annuity contracts whereby they obtain a higher than the (safe) market interest rate if they survive, in return for losing all the money at death; a "bank" on the other side of this transaction would then on average, if the contract is written with many individuals, obtain a safe return. The perpetual-youth model inherits some of the characteristics of the OG model but, at the same time, is a move toward a dynasty model: as the probability of survival approaches one, the model's features approach those of the pure dynasty setting.

[18]Note also that a planner's objective in the OG setting would need to involve a social welfare function across cohorts.

Warm glow One can also append an OG model with *bequest functions*. The idea here is that agents are not altruistic per se, but derive utility from the act of giving itself (typically, to their children). This setting is referred to as one of *warm glow*: giving makes one happy (glow). To illustrate, consider the two-period life OG model. The preferences are now

$$u(c_y) + \beta u(c_o) + \varphi(b'),$$

where φ (like u) is an increasing, strictly concave function and b' is the amount of bequests given. Thus, the budget constraints for cohort t (assuming a stationary endowment economy for simplicity) read

$$c_{y,t} + q_t a_{t+1} = \omega_y + b_t \text{ and } c_{o,t+1} + b_{t+1} = \omega_o + a_{t+1}.$$

Here, b_t is the bequest inherited from cohort $t-1$ (the parents) and b_{t+1} is the amount bequeathed to cohort $t+1$. Market clearing could again be expressed by requiring that total consumption equal total endowments period by period; equivalently, one could simply require a_{t+1} to be zero at all times. Note that the old at time 0 face a meaningful intertemporal tradeoff: maximize, by choice of $(c_{o,0}, b_0)$, $\beta u(c_{o,0}) + \varphi(b_0)$ subject to $c_{o,0} + b_0 = \omega_o$.

An alternative interpretation of this function is that the consumer derives utility from their end wealth: "dying with a bigger bank account balance" gives you higher utility. Regardless of interpretation, this kind of model can be seen as a *behavioral* model: agents no longer have preferences over consumption goods (and leisure) but are endowed with utility functions defined by the act of giving. A conceptually different approach is to assume that parents value their offspring's consumption paths; if they do, but they do so in a way that is inconsistent with the way the children value their own consumption paths, such preferences are said to capture impure altruism. A model with pure altruism is instead one where the parent cares about the child *the way the child cares about themselves*, in which case one can no longer label the model behavioral since now preferences are defined over goods. If this is true for all cohorts, we can specify the utility of cohort t as follows:

$$u(c_{y,t}) + \beta u(c_{o,t+1}) + \tilde{\beta}\varphi(b_{t+1}),$$

where

$$\varphi(b_{t+1}) = u(c_{y,t+1}) + \beta u(c_{o,t+2}) + \tilde{\beta}\varphi(b_{t+2}),$$

and so on. Notice here that $\tilde{\beta}$, the weight on the child's indirect utility, does not have to equal β. This reveals a recursive structure, which is most easily described with dynamic programming notation. Assuming a constant interest rate $1/q$ for simplicity, we obtain

$$\varphi(b) = \max_{c_y, c_o, a', b'} u(c_y) + \beta u(c_o) + \tilde{\beta}\varphi(b'),$$

subject to $c_y + qa' = \omega_y + b$ and $c_o + b' = \omega_o + a'$.

In sum, we have seen the perpetual-youth model, which looks more and more like the dynastic model as the survival probability goes to one. The warm glow model, on the other hand, can be seen as a behavioral OG model, except in the very special case where φ is (a constant times) the indirect utility function of the next cohort. Then, the model becomes a *dynastic life-cycle model*: one where there are life-cycle features but where parents care about children in a (purely) altruistic way. In this very special case, thus, the basic long-run features are exactly those of the simpler dynasty model we use in our benchmark. The long-run interest rate is now given by $1/\tilde{\beta}$.