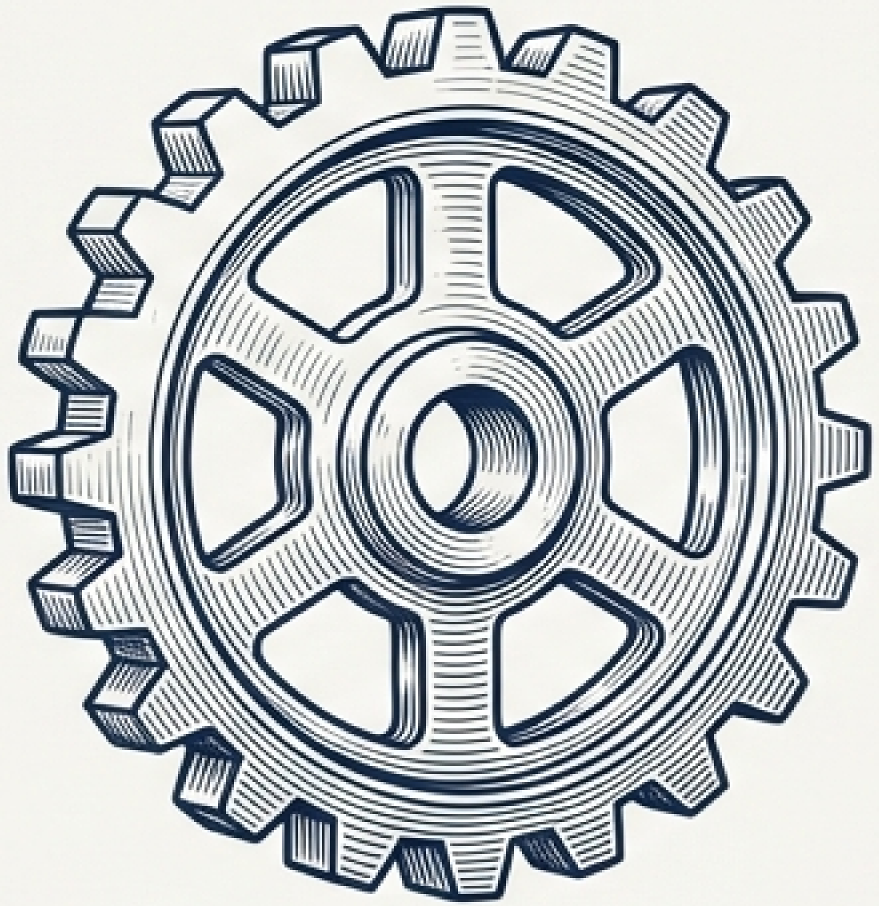
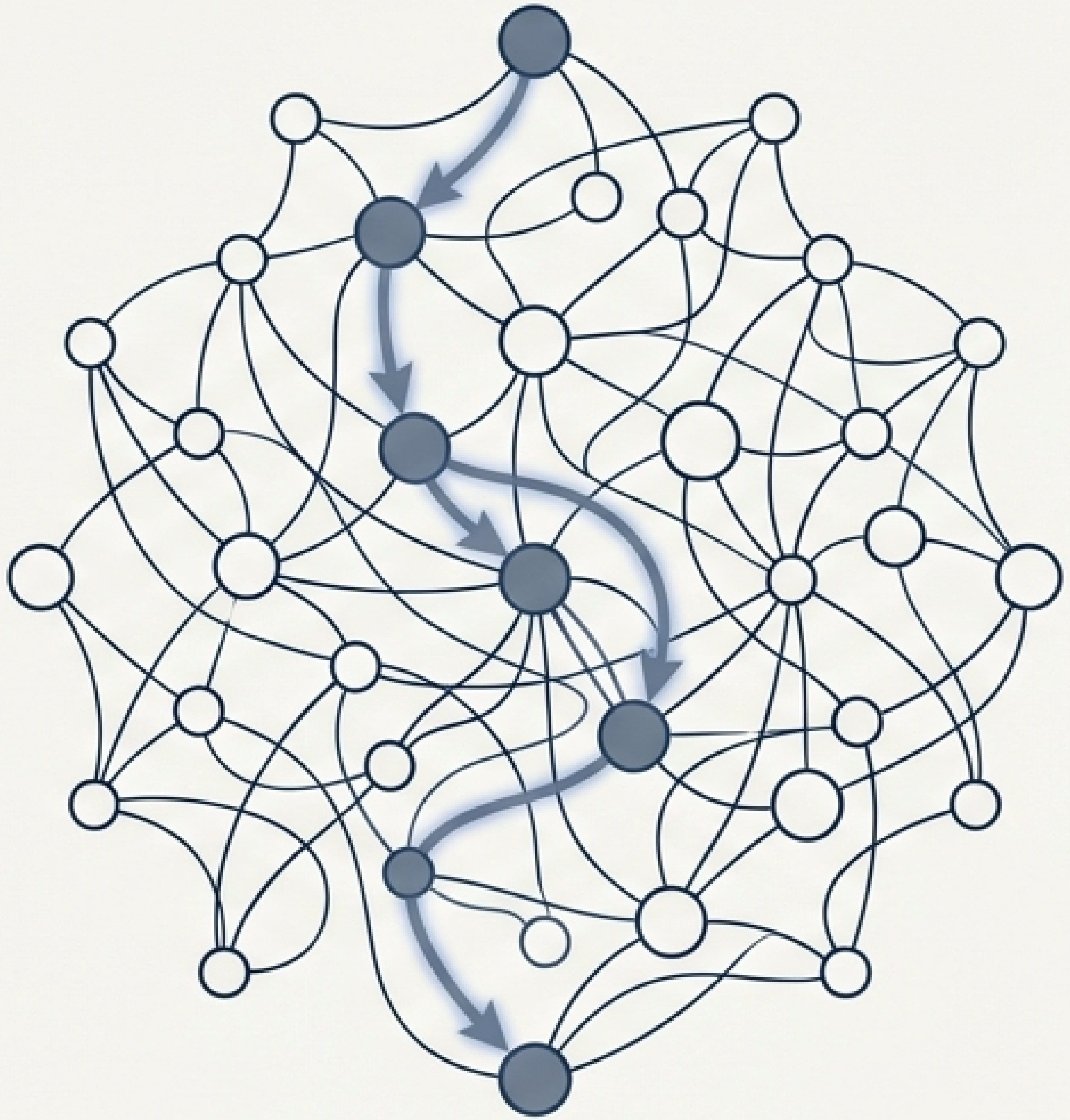
Dynamic Optimization

The Micro-Foundations of Macroeconomic Growth



Solow Model (Exogenous)

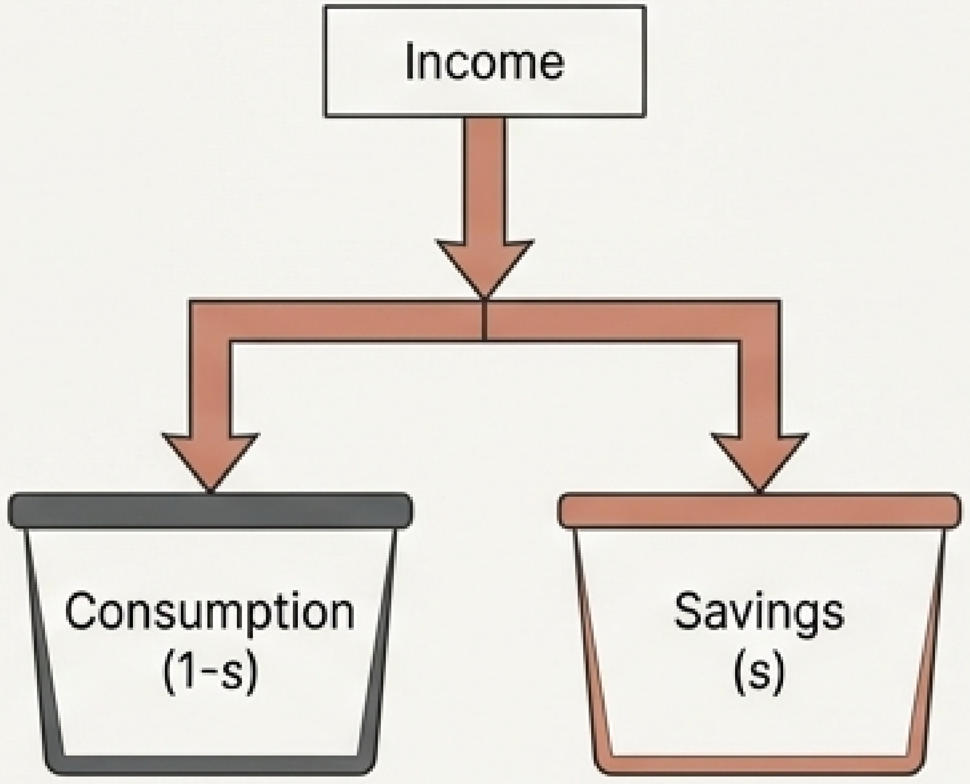


Ramsey-Cass-Koopmans (Endogenous)

The Shift: Moving beyond describing *how* capital evolves to explaining ***why*** agents save and invest.
The Goal: Deriving the "optimizing neoclassical growth model"—the backbone of modern policy analysis and Real Business Cycle theory.

The Limitation of Constant Savings Rates

The Solow Standard

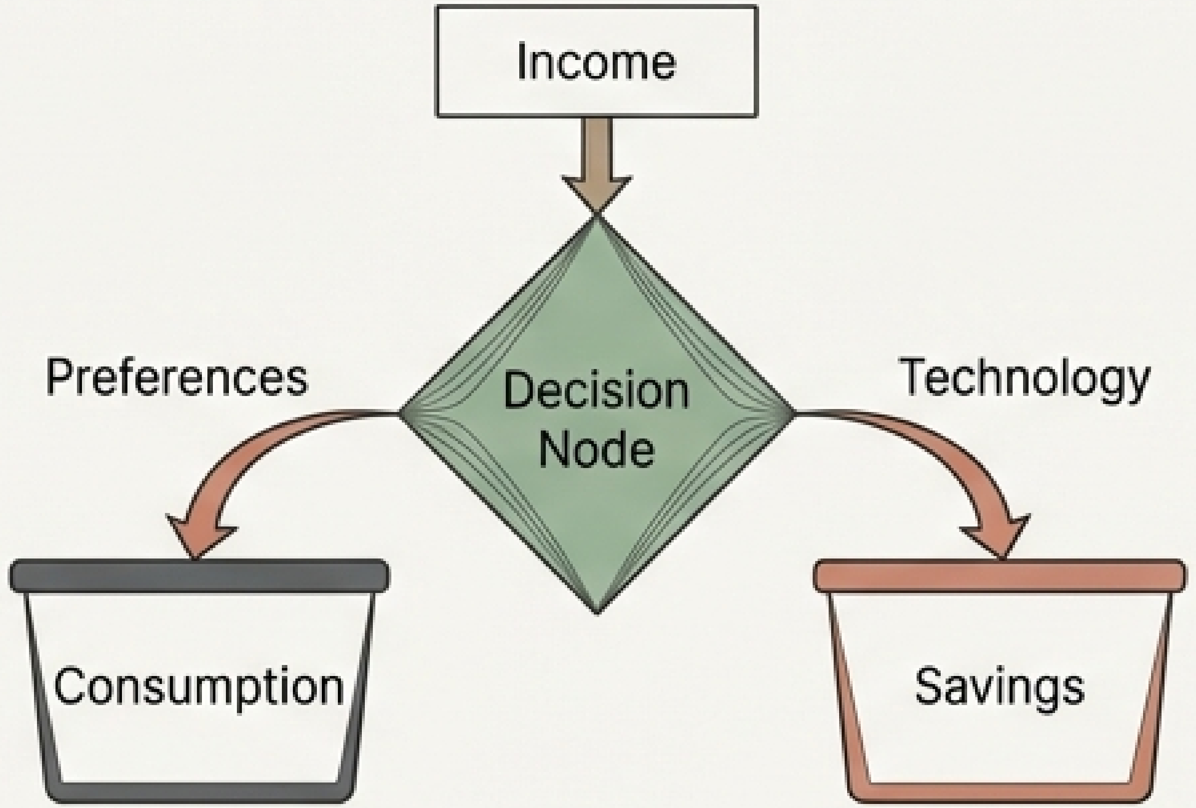


- Assumes agents save a constant proportion s of income.
- Capital accumulation is mechanical:

$$k_{t+1} = (1-\delta)k_t + sf(k_t)$$

- **The Gap:** Policy interventions cannot affect the savings rate, only the capital level. Lacks behavioral response.

The Optimizing Solution



- Introduced by Cass (1965) and Koopmans (1963).
- Saving s is endogenously determined by inter-temporal choice.
- Agents respond to shocks (productivity, taxes) by smoothing consumption.

Assumption: Agents are fully rational and forward-looking—a "first approximation" introduced by Milton Friedman (1957).

The Representative Agent & The Trade-off

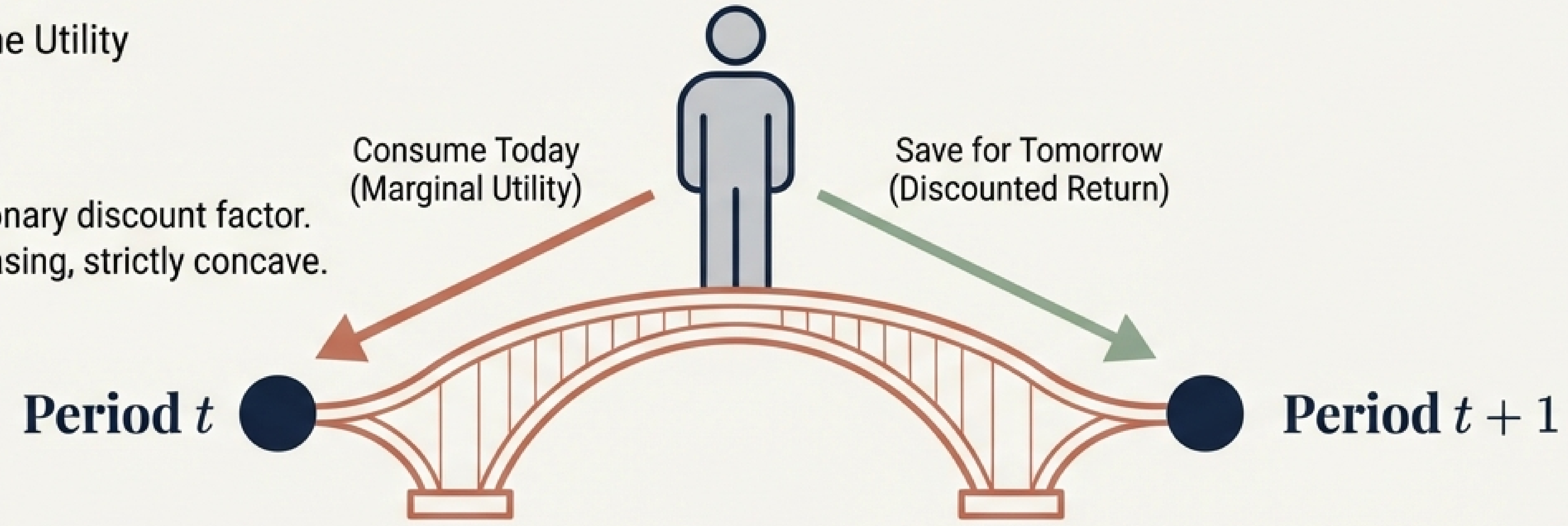
The Objective

Maximize Lifetime Utility

$$U = \sum_{t=0}^{T} \beta^t u(c_t)$$

$\beta = 1/(1+\rho)$: Stationary discount factor.

$u(c)$: Strictly increasing, strictly concave.



The Constraint

Resource Constraint

$$c_t + a_{t+1} = w_t + (1 + r_t)a_t$$

Consumption + Saving = Income + Assets

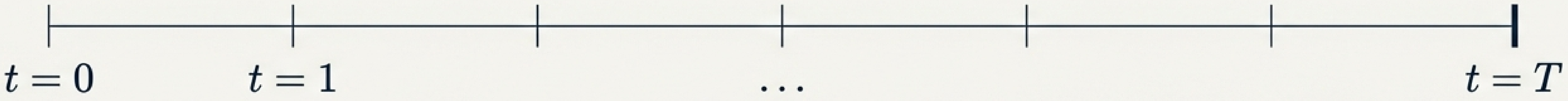
The Decision

The Core Mechanism

Sacrifice consumption today ($u'(c_t)$ cost) for consumption tomorrow.

Toolkit A: Sequential Methods (Finite Horizon)

Solving for a specific sequence of allocations $\{c_t, k_{t+1}\}$ over a fixed timeline T.



$$\mathcal{L} = \sum_{t=0}^{T} \beta^t u(c_t) + \sum_{t=0}^{T} \mu_t [w_t + (1+r_t)a_t - c_t - a_{t+1}]$$

1. First Order Conditions (FOCs)

Derivatives w.r.t. c_t and a_{t+1} set to zero. Optimizes the path between start and finish.

2. Complementary Slackness

$$\mu_t [w_t - c_t] = 0$$

Ensures constraints are respected (or non-binding).

3. Terminal Condition

$$a_{T+1} = 0$$

Insight: Agents leave no assets on the table at the end of the world. Start with zero, end with zero.

The Euler Equation: Marginal Cost vs. Benefit

$$u'(c_t) = \beta u'(c_{t+1})(1 + r_{t+1})$$

Marginal Cost

The pain of reducing consumption today.

Discounted Utility

The value of future consumption, discounted by impatience (β).

Market Return

The yield on savings. How many units you get back.

Consumption Smoothing Logic:

- If Cost > Benefit: Consume more today.
- If Cost < Benefit: Save more for tomorrow.
- If $\beta(1 + r) = 1$: Consumption is constant.

The Neoclassical Growth Model (NGM)

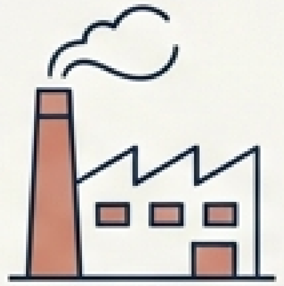
Transitioning from an endowment economy to a production economy.

The Planner's Problem

Maximize $\sum \beta^t u(c_t)$

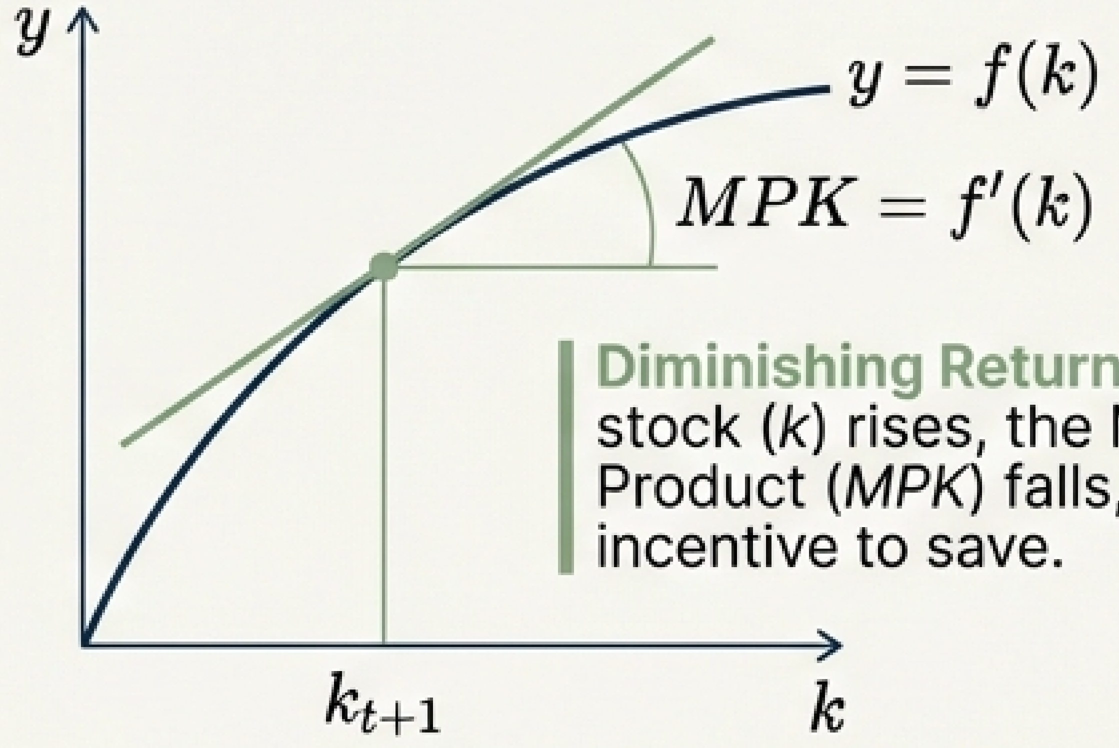
subject to:

$$c_t + k_{t+1} = f(k_t) + (1 - \delta)k_t$$



The New Euler Equation

$$u'(c_t) = \beta u'(c_{t+1}) \underbrace{[f'(k_{t+1}) + 1 - \delta]}_{\text{Return on Capital}}$$



Diminishing Returns: As capital stock (*k*) rises, the Marginal Product (*MPK*) falls, lowering the incentive to save.

Scaling to Infinity: Constraints & Transversality

When $T \to \infty$, we need new rules to prevent gaming the system.

No-Ponzi Game (NPG)

Institutional Constraint

$$\lim_{T\to\infty} \frac{a_{T+1}}{(1+r)^T} \geq 0$$

You cannot roll over debt forever. Lenders will eventually stop lending.

Prevents dying in infinite debt.

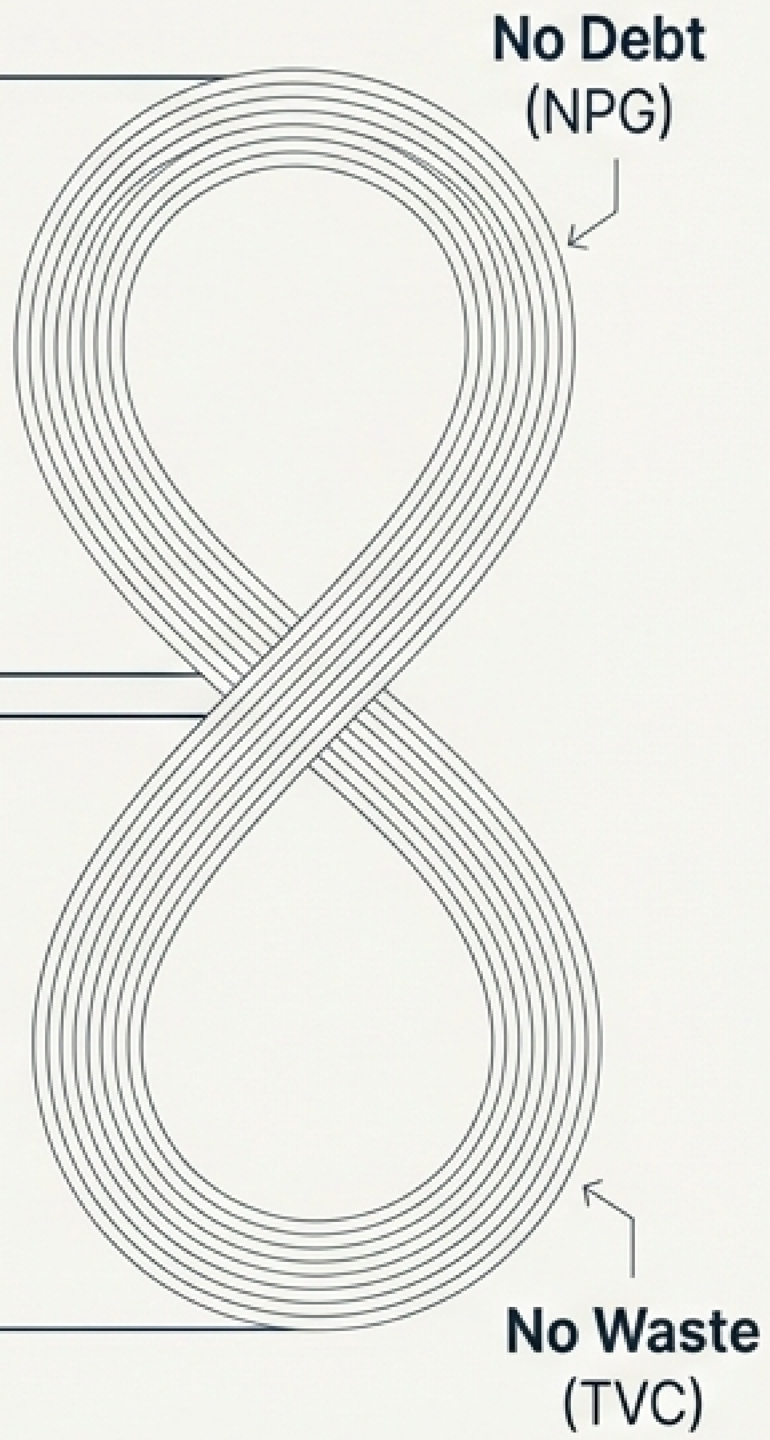
Transversality Condition (TVC)

Optimality Condition

$$\lim_{t\to\infty} \beta^t u'(c_t) k_{t+1} = 0$$

You should not accumulate infinite assets you never consume. The ‘shadow value’ of capital must go to zero.

Prevents dying with unspent wealth.



Explicit Solutions: The 'Log-Cobb-Full' Case

A rare analytical solution that proves the theory works.

The Assumptions

Log Utility:
$u(c) = \log c$

Cobb-Douglas:
$f(k) = Ak^{\alpha}$

Full Depreciation:
$\delta = 1$

The Mathematical Result

Optimal Policy Rule:

$$k_{t+1} = \alpha\beta A k_t^{\alpha}$$

A rare case with a closed-form solution.

The Insight

Implied Savings Rate:

$$s = \alpha\beta$$

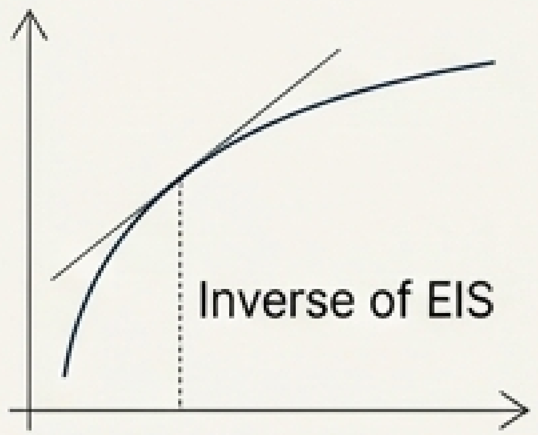
In this specific case, the optimal savings rate is constant—just like Solow. However, it is now determined by parameters of patience (β) and technology (α).

Balanced Growth & CRRA Utility

Consistent with Kaldor's facts, growth rates must be constant in the long run.
The Solution: **Constant Relative Risk Aversion (CRRA) Utility.**

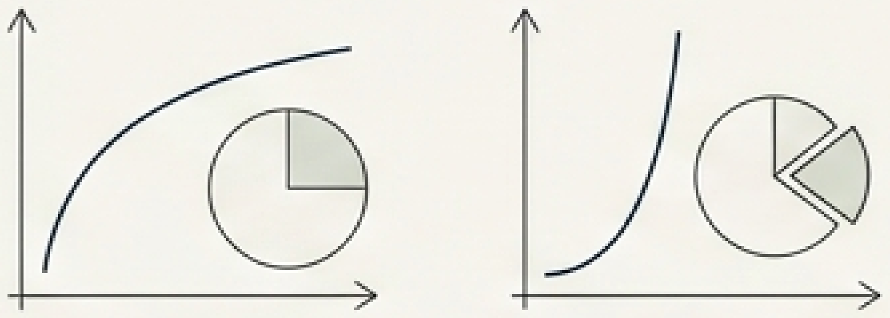
$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}$$

1. The Parameter σ



Inverse of Intertemporal Substitution ($EIS = 1/\sigma$). Measures curvature.

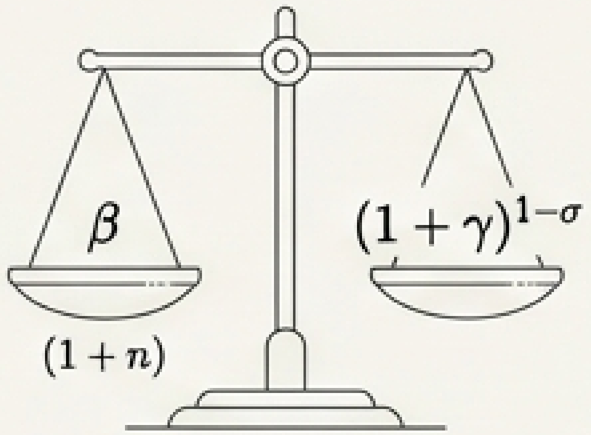
2. The Cases



$\sigma \to 1$: Log Utility (Constant shares).

$\sigma > 1$: Income effect dominates (Savings vary).

3. The Adjusted Discount



With population growth (n) and tech growth (γ):

$\tilde{\beta} = \beta(1+n)(1+\gamma)^{1-\sigma}$

Toolkit B: Recursive Methods

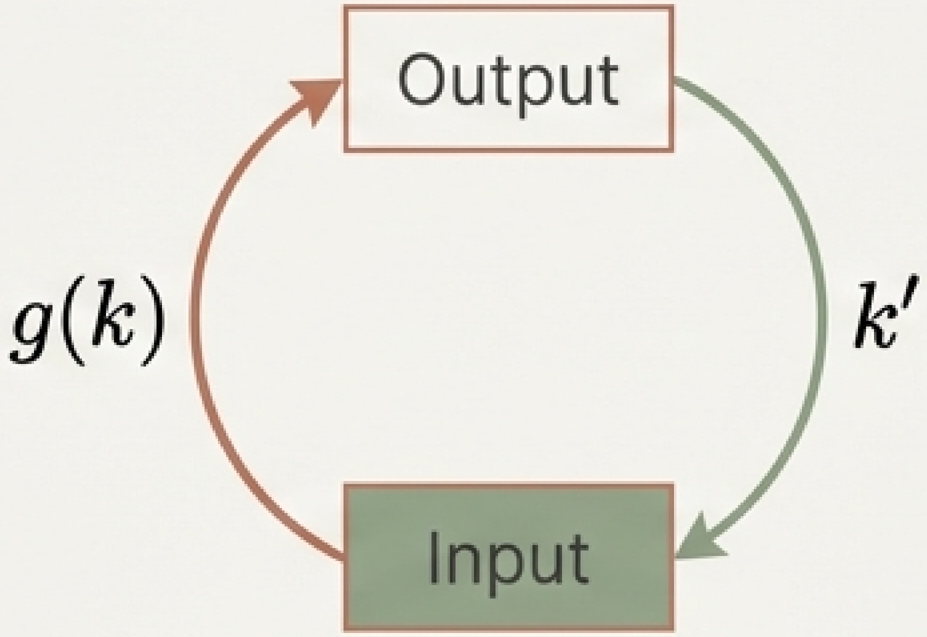
A philosophical shift: From Sequences to Functions.

Sequential View

The Path $\{k_{t+1}\}_{t=0}^{\infty}$

Planning every move of a chess game in advance.
History Dependent.

Recursive View



The Rule $k' = g(k)$

Having a strategy for any board position.
State Dependent.

Stationarity: If the problem looks the same at time t as at time $t + 1$, time doesn't matter—only the **State Variable** (k) matters. Notation change: Drop t. Use x (today) and x' (tomorrow).

The Bellman Equation

The engine of Dynamic Programming.

$$V(k) = \max_{k'} \{ \underbrace{u(f(k) + (1-\delta)k - k')}_{\text{Current Payoff}} + \underbrace{\beta V(k')}_{\text{Continuation Value}} \}$$

$V(k)$ **(Value Function):** Max lifetime utility attainable starting from state k.

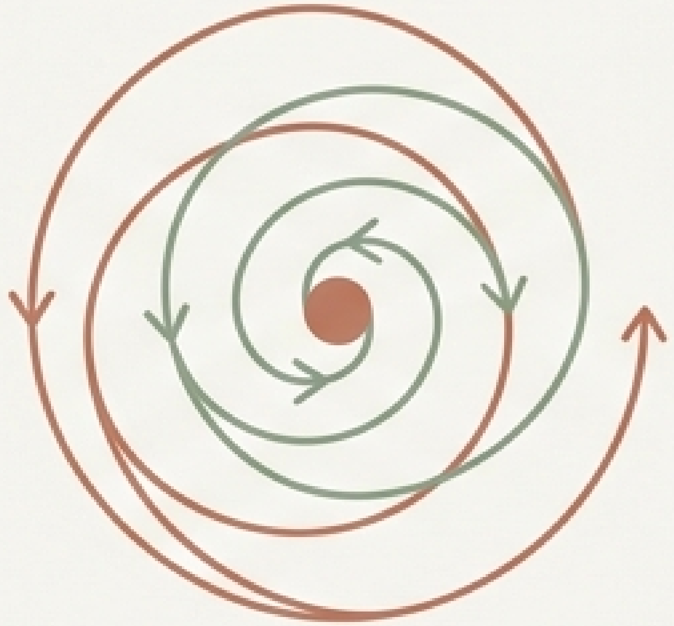
Current Payoff: Utility from consumption today.

Continuation Value: The discounted value of starting next period with capital k'.

The Policy Function: $k' = g(k)$ is the rule that maximizes the Right-Hand Side.

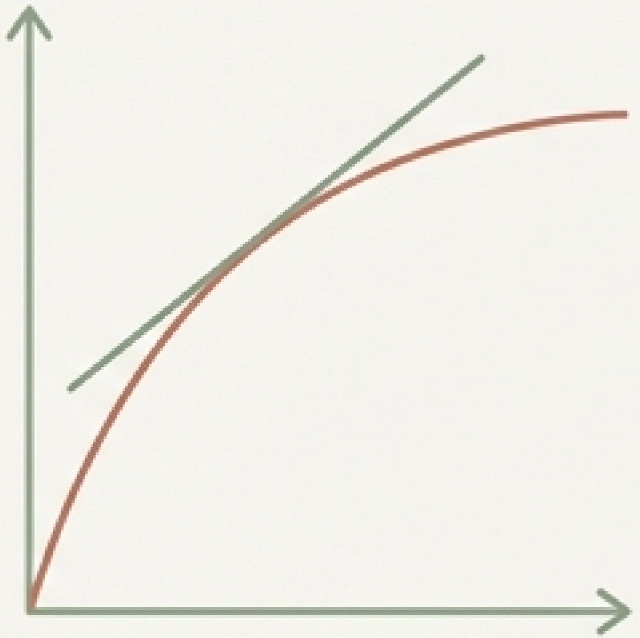
Why We Trust the Math: Value Function Properties

Contraction Mapping



The Bellman operator is a contraction. We can find $V(k)$ by iteration. Start with a guess V_0, iterate, and it is **guaranteed** to converge to the true V.

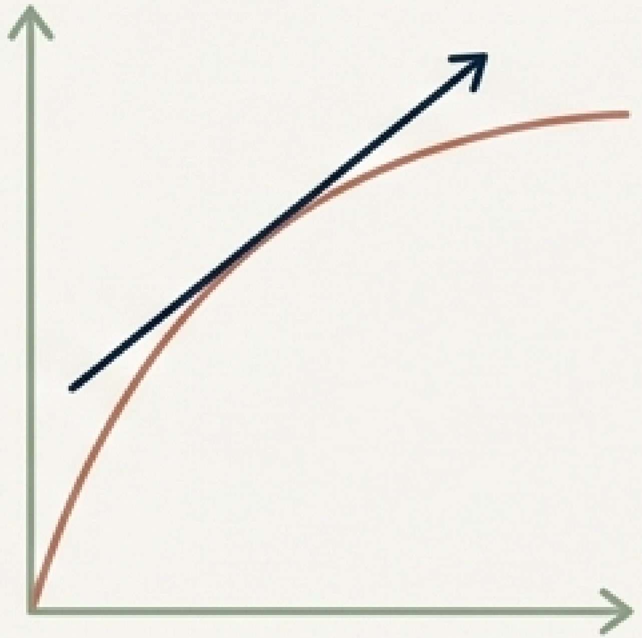
Shape Properties



Strictly Increasing: More capital is always better.

Strictly Concave: Diminishing returns apply to lifetime value.

Differentiability



Allows us to use the **Envelope Theorem** to solve. Under

standard assumptions, there is only one true Value Function.

Solving Recursively: The Functional Euler Equation

Step 1: The Goal

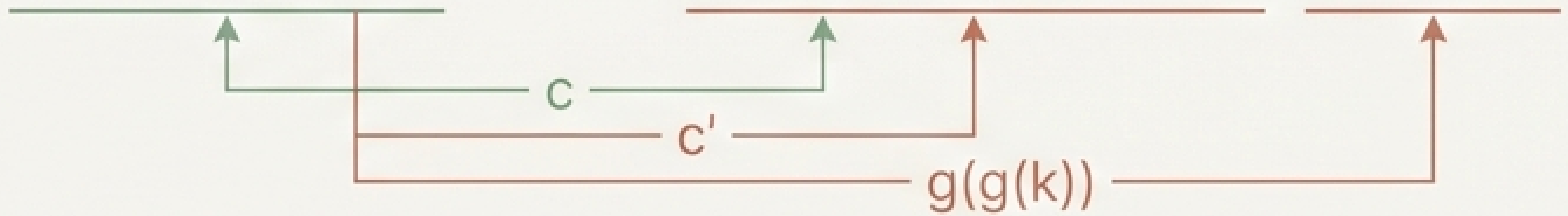
Find the policy function $g(k)$.

Step 2: The Functional Euler Equation

$$u'(c) = \beta u'(c')\,[f'(k') + 1 - \delta]$$

Becomes:

$$u'(f(k) - g(k)) = \beta u'\big(f(g(k)) - g(g(k))\big) R(g(k))$$



Step 3: Guess & Verify Technique (Visual Flow)

1. Guess form of Value Function:

$$V(k) = a + b \ln k$$

2. Maximize the Bellman RHS.

$$\max_{RHS} (V(k) + b \ln k)$$

3. Match Coefficients to verify the guess.

Verify parameters a and b must satisfy.

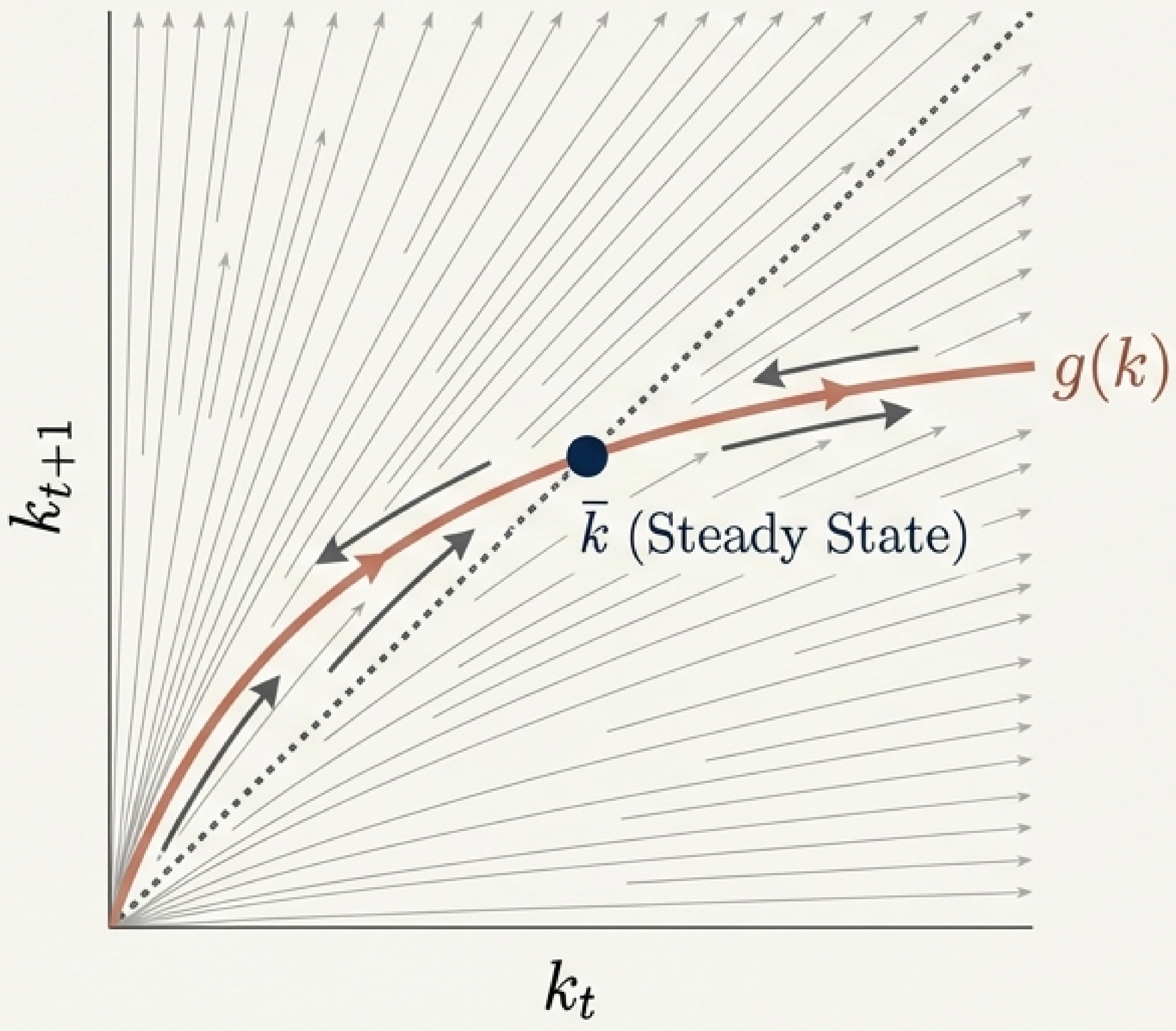
Result:

Result:

Confirms $s = \alpha\beta$ (for Log-Cobb case).

Global Stability & Convergence Dynamics

The Phase Diagram



Calibration & Speed

How fast do we reach steady state?

Depends on σ (Utility Curvature).

- **High σ (Leontief-like):** Agents hate change → **Slow Convergence**
- **Low σ (Linear):** Agents don't care about smoothing → **Fast Convergence**

Empirical Reality:

Macro Standard: $\sigma \approx 1$ (Log utility) or 2.

Estimates: $1/\sigma \in [0.1, 0.8]$ (Hall, Attanasio & Weber).

Academic Editorial

Synthesis: Two Toolkits, One Framework

Feature	**Sequential Methods**	**Recursive Methods**
Focus	Paths / Sequences (c_t)	Rules / Functions ($g(k)$)
Tool	Lagrangian / Sums $\sum_{k=1}^{n} L_l$	Bellman Equation $V(k) = \max_{k}\{\dots\}$
Constraint	Transversality Condition $\lim_{k\to\infty}(t-0)$	Contraction Mapping
Best For	Theoretical Proofs, Simple Examples	Numerical Simulation, Complex Stochastic Models

The Final Takeaway: By endogenizing savings, the Optimizing NGM allows us to analyze how policy alters the *incentives* to accumulate wealth, providing the micro-foundations for modern macroeconomics.