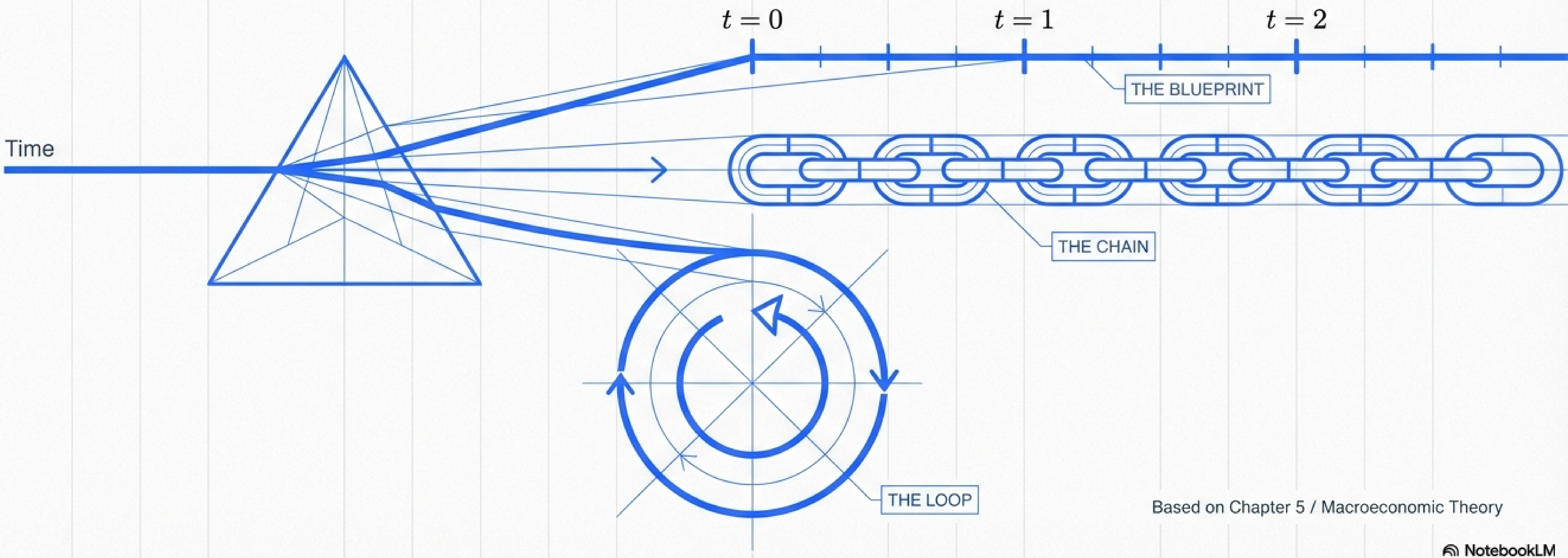
Dynamic Competitive Equilibrium

The Architectures of Time, Trade, and Optimization



Based on Chapter 5 / Macroeconomic Theory

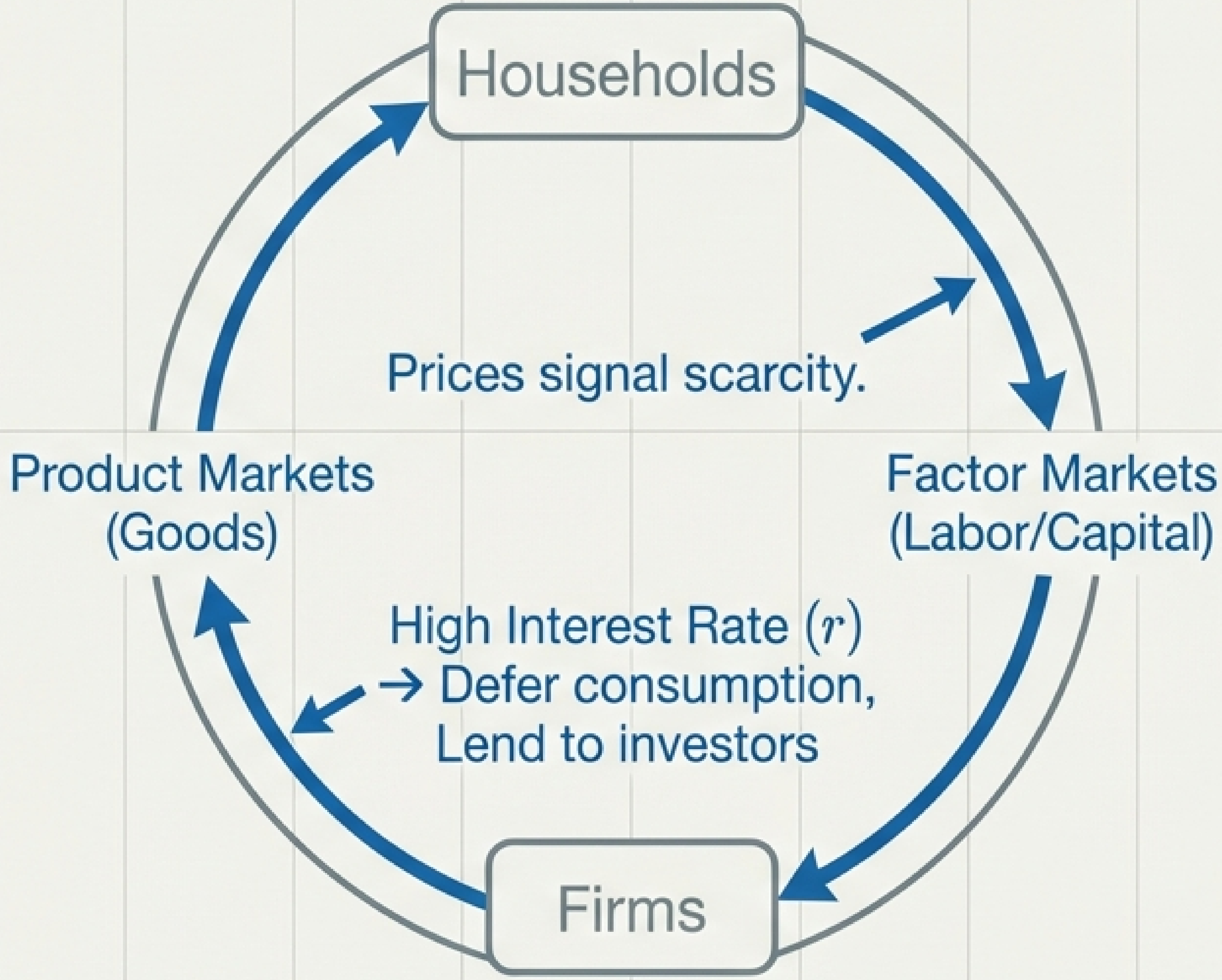
The Macroeconomic Benchmark

Perfect Competition as the Physics of a Vacuum

Before analyzing market failures, we define the frictionless ideal. In this vacuum, private incentives steer production and consumption perfectly via the price mechanism.

The Agents

- Representative Households: Utility Maximizers. Supply labor/capital, demand goods.
- Representative Firms: Profit Maximizers. Hire inputs until Marginal Cost = Marginal Return ($MPL = w$, $MPK = r$).



Two Conditions for Equilibrium

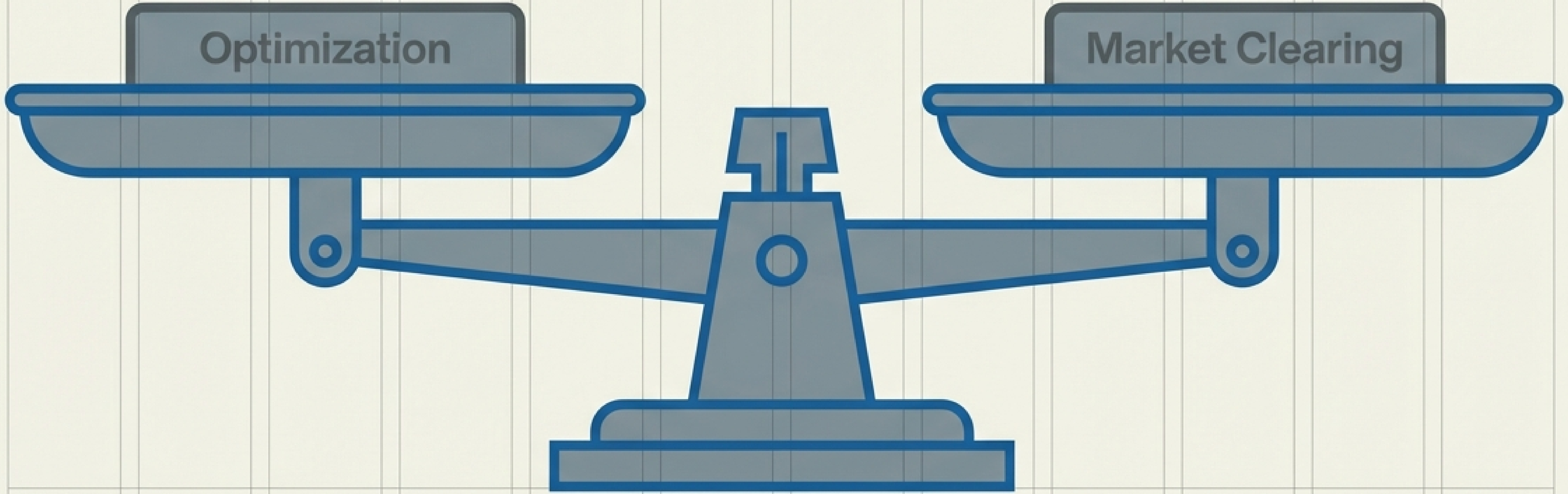
Agents maximize objectives taking prices as given.

Variables not chosen are treated as constants.

Resource Feasibility

$$\text{Demand} = \text{Supply}$$

Applies to all traded goods and assets.

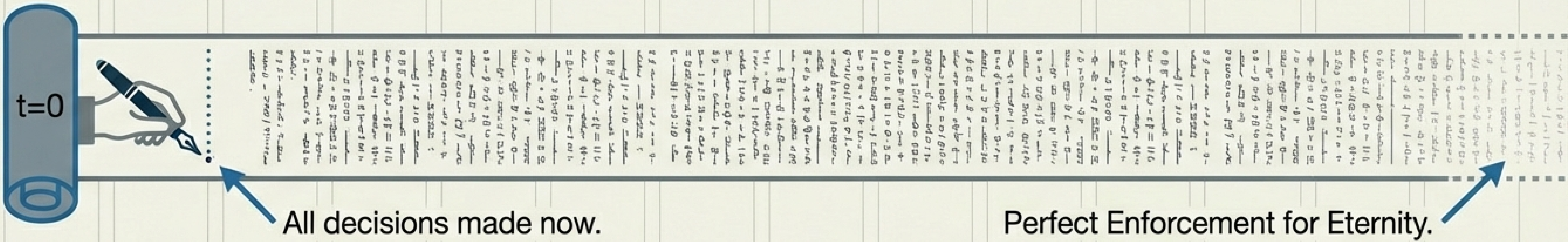


Result: Pareto Optimality (in the absence of frictions).

Time Horizon: Infinite. We assume a continuum of time to remove “end of time” distortions.

Lens I: Arrow-Debreu (The Blueprint)

The Date-0 Grand Contract



$$\sum_{t=0}^{\infty} p_t c_t = \sum_{t=0}^{\infty} p_t y_t$$

Lifetime summation.

Lifetime summation.

Price paid at $t = 0$ for delivery at t.

Present value of total income.

Pricing Scarcity and Impatience

Application: The Endowment Economy

In a pure harvest economy (no production), prices reflect two forces: how impatient we are, and how scarce goods are.

Result: Perfect Consumption Smoothing. If aggregate endowment is constant, individual consumption is constant.

$$p_t = \beta^t \frac{u'(c_t)}{u'(c_0)}$$

(Note: p_0 is normalized to 1)

Discounting. The price declines because agents are impatient.

Scarcity. If endowment (Y_t) is low → Marginal Utility is high → Price is high.

The Neoclassical Growth Engine

Application: Production with Capital and Labor

The Firm

Maximizes static profit.

$MPL = w_t$ (Hire labor until cost equals output)
$MPK = r_t$ (Hire capital until rental rate equals return)

The Household

Invests to smooth consumption.

Action: Balances consumption vs. investment (i_t).

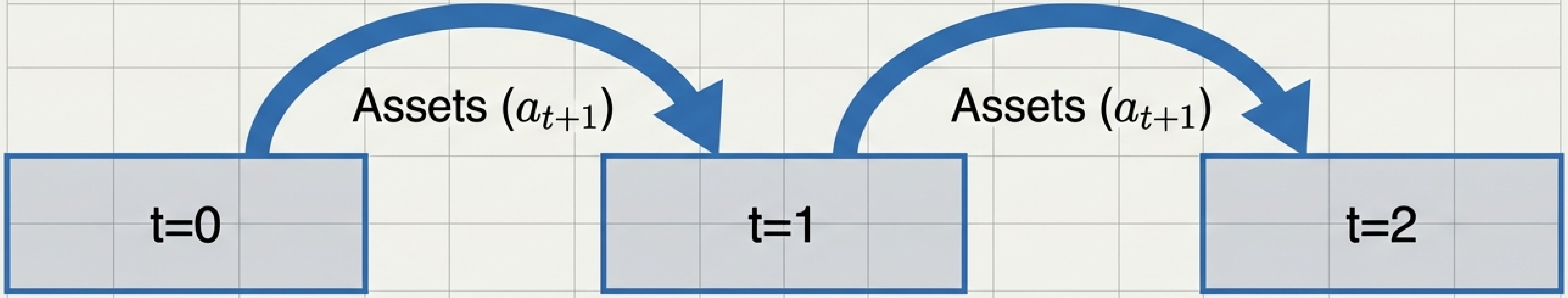
The Euler Equation for Capital

$$p_t = p_{t+1}[r_{t+1} + (1 - \delta)]$$

The cost of buying capital today (p_t) = The discounted value of renting it tomorrow (r_{t+1}) + selling the scrap ($1 - \delta$).

Lens II: Sequential Equilibrium (The Process)

Step-by-Step Trading



Markets reopen **every period**. Agents hold assets to link the present to the future.

The Period Budget Constraint

$$c_t + q_t\, a_{t+1} = y_t + a_t$$

- c_t: Consumption today.
- q_t: Price of a bond (inverse of interest rate).
- a_{t+1}: Savings carried to tomorrow.
- $y_t + a_t$: Resources available today.

Asset Market Clearing

$$\int a_i \, di = 0$$

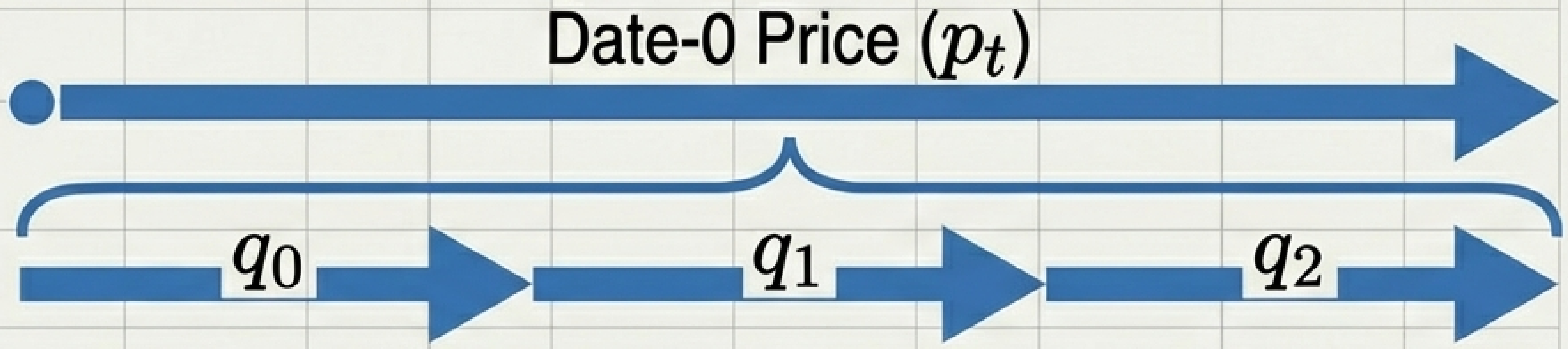
(For every borrower, there is a lender).

The Great Equivalence

Different Lenses, Identical Outcomes

Track 1 (Arrow-Debreu):

Track 2 (Sequential):



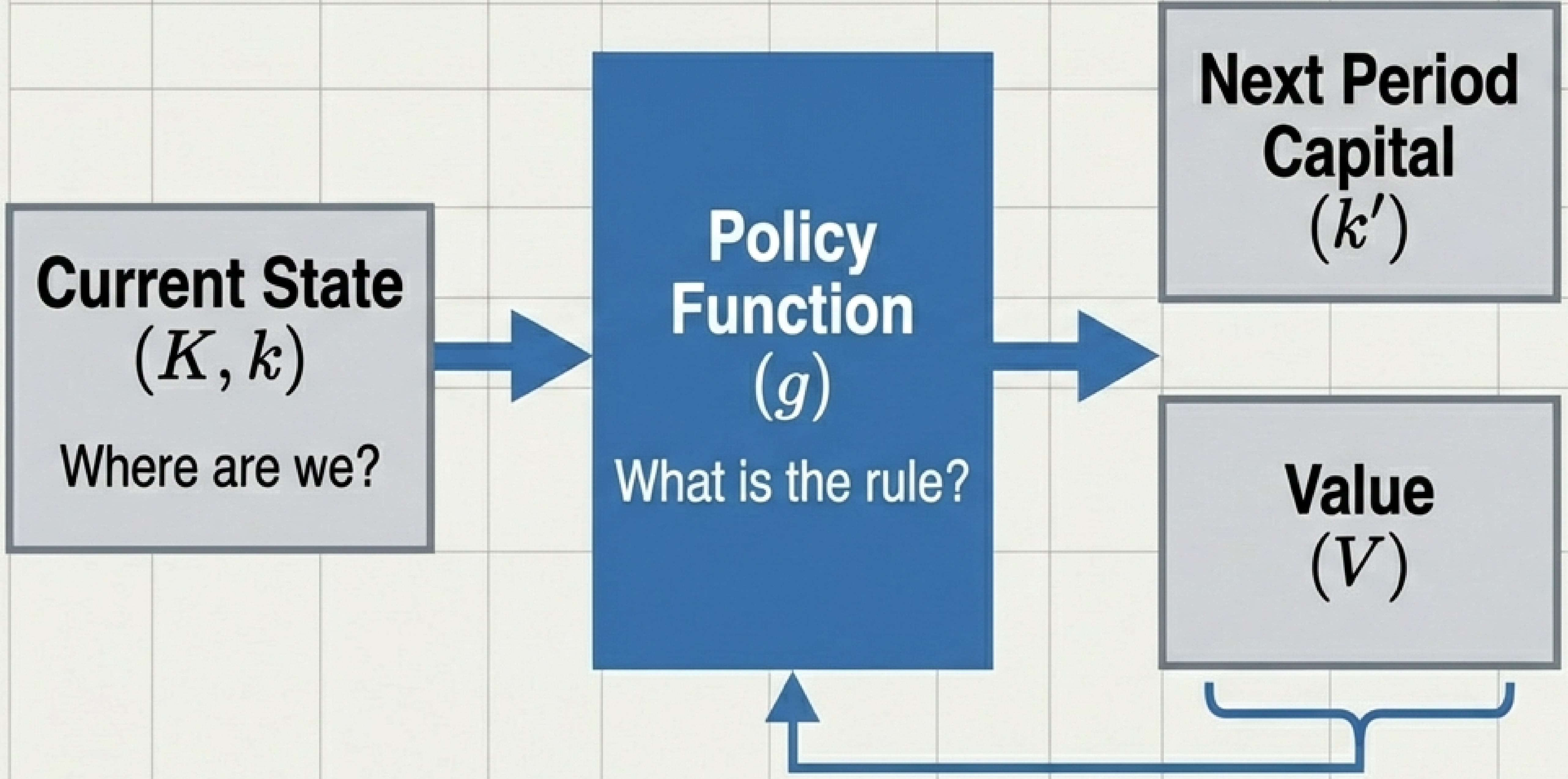
$$p_t = q_0 \times q_1 \times \ldots \times q_{t-1}$$

The Arrow-Debreu price is the **cumulative product** of all sequential asset prices.

- Arrow-Debreu Price ≈ Present Value (Discounted to $t = 0$).
- Sequential Interest Rate ≈ Flow Price (Spot rate between t and $t + 1$).

Lens III: Recursive Equilibrium (The Machine)

From Infinite Sequences to Functional Rules



Key Definitions

- **State Variable** (K): Sufficient statistic for history.
- **Policy Function** ($k' = g(k, K)$): The decision rule.
- **Value Function** ($V(k, K)$): The lifetime utility of being in this state.

The Consistency Loop

Rational Expectations in a Recursive World

Fixed Point

The Agent's Problem

$\mathrm{Max}\, V(k, K)$ subject to constraints.

The agent needs to predict prices (r, w), which depend on Aggregate Capital (K').

The agent *assumes* a Law of Motion:

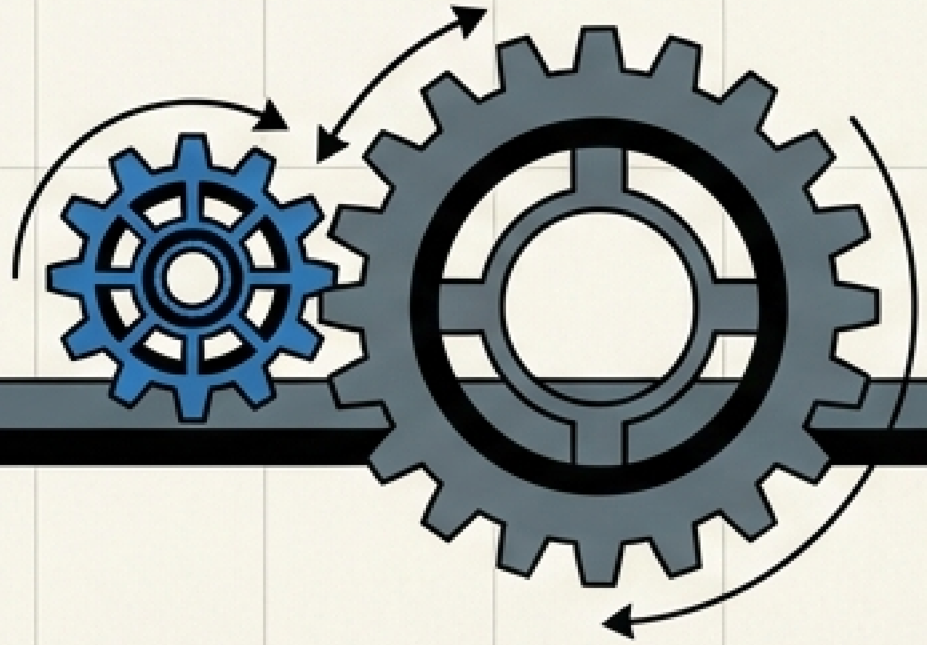
$$K' = G(K)$$

The Equilibrium Condition

Consistency

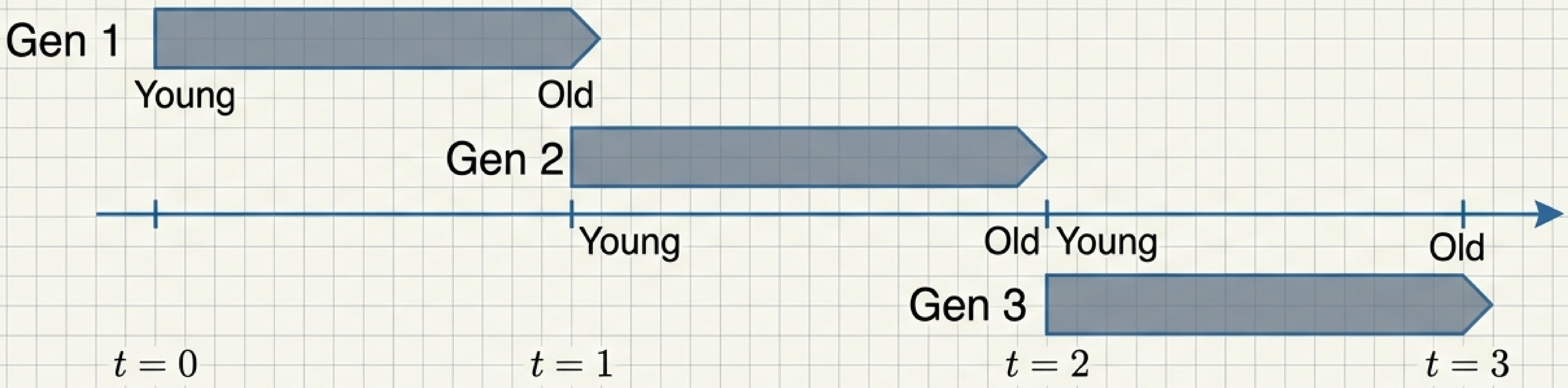
$$G(K) = g(K, K)$$

The aggregate law of motion $G(K)$ must equal the individual policy $g(k, K)$ when the individual holds the average capital $(k = K)$.



The Demographic Twist: Overlapping Generations

From Infinite Dynasties to Finite Lives



- Agents save for **their own** retirement, not for eternity.
- Market Clearing: The Young buy assets from the Old. There is no "net" saving in a pure endowment economy—intergenerational trade must sum to zero.

The Inefficiency of Finite Time

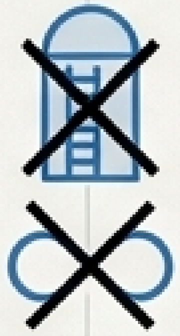
Autarky and Dynamic Inefficiency

Scenario

Endowment is high when Young, low when Old.
Everyone wants to save.

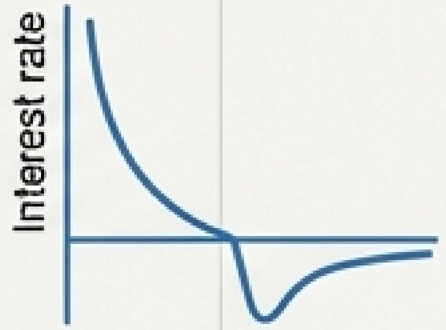
The Friction

- **No** storage technology.
- **No infinite** future to borrow from.



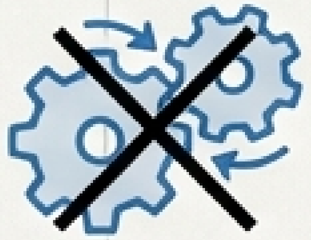
The Result

If demand for savings is high, the interest rate can plummet.



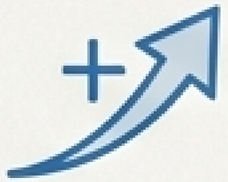
Possibility of $q > 1$
(Negative Real Interest Rates)

Unlike Dynastic models, OG **equilibria** might **not** be Pareto Optimal.



Implication

A government transfer (Social Security) from Young to Old can **strictly improve welfare.**



NotebookLM

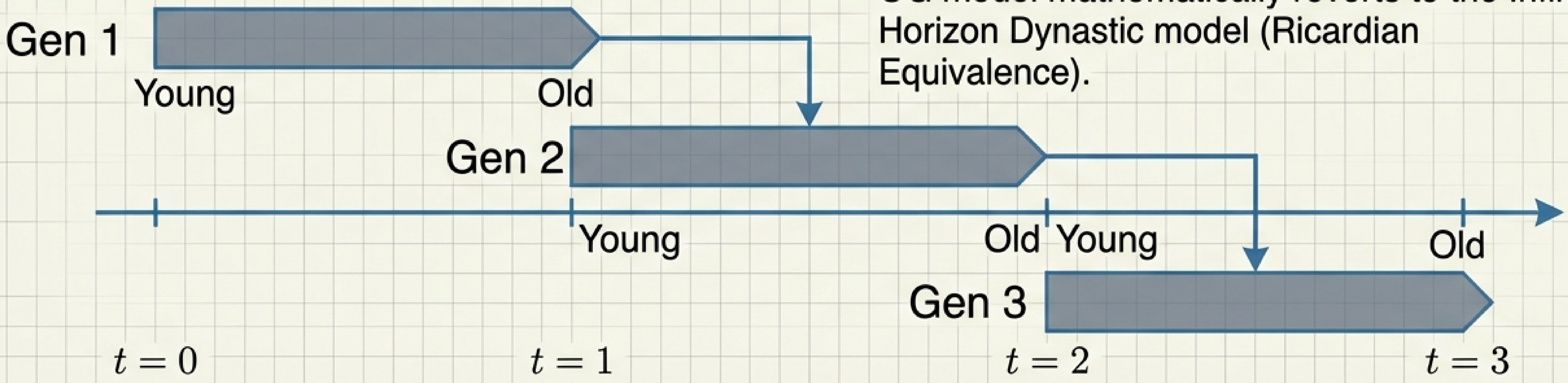
Restoring the Dynasty

Capital, Bequests, and Altruism

OG with Production

Young work & save (buy Capital K). Old rent K to firms, consume return.

Note: Solved forward. History matters.



Variations

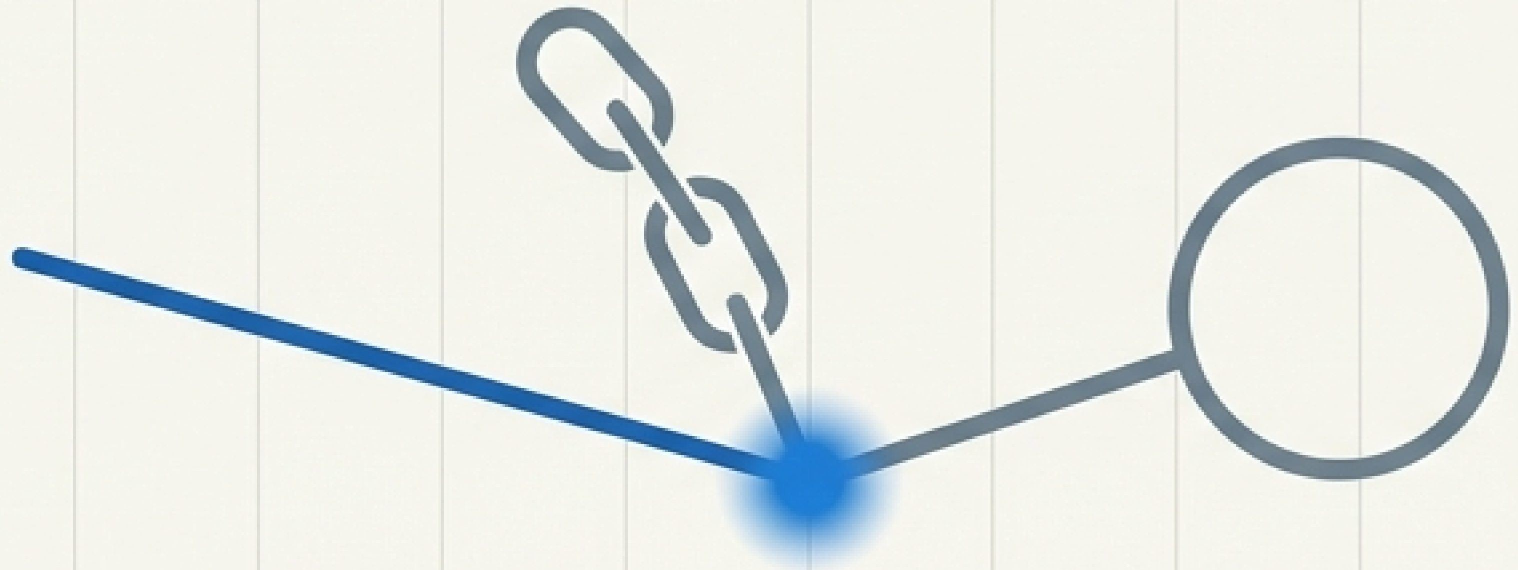
1. **Warm Glow**: Utility from the act of giving ($U(c) + \phi(b)$).
2. **Pure Altruism**: Caring about children's utility.
   - Result: If parents care enough, the finite-horizon OG model mathematically reverts to the Infinite Horizon Dynastic model (Ricardian Equivalence).

Summary Matrix: Architectures of Time

Feature	Arrow-Debreu	Sequential	Recursive	Overlapping Gen (OG)
Time Concept	Static (Date-0)	Period-by-Period	Infinite/State-Based	Finite Lives
Solution Object	Sequence (c_t)	Sequence (c_t, a_t)	Functions (V, g)	History Sequence
Key Feature	The Grand Contract	Flow Constraints	Consistency $G = g$	Life-Cycle Saving
Primary Use	Theoretical Proofs	Finance/Asset Pricing	Computation/Macro	Fiscal Policy/Pensions

The Mathematical Invisible Hand

Synthesis



Whether through a single contract, daily trades, or functional rules, the Invisible Hand is mathematically defined by two invariant forces:

Optimization

Individual agents doing the best they can. $\max U(c)$).

Consistency

Aggregate results validating individual expectations. ($G = g$).

These architectures allow us to simulate history before it happens.