Uncertainty in Macroeconomics

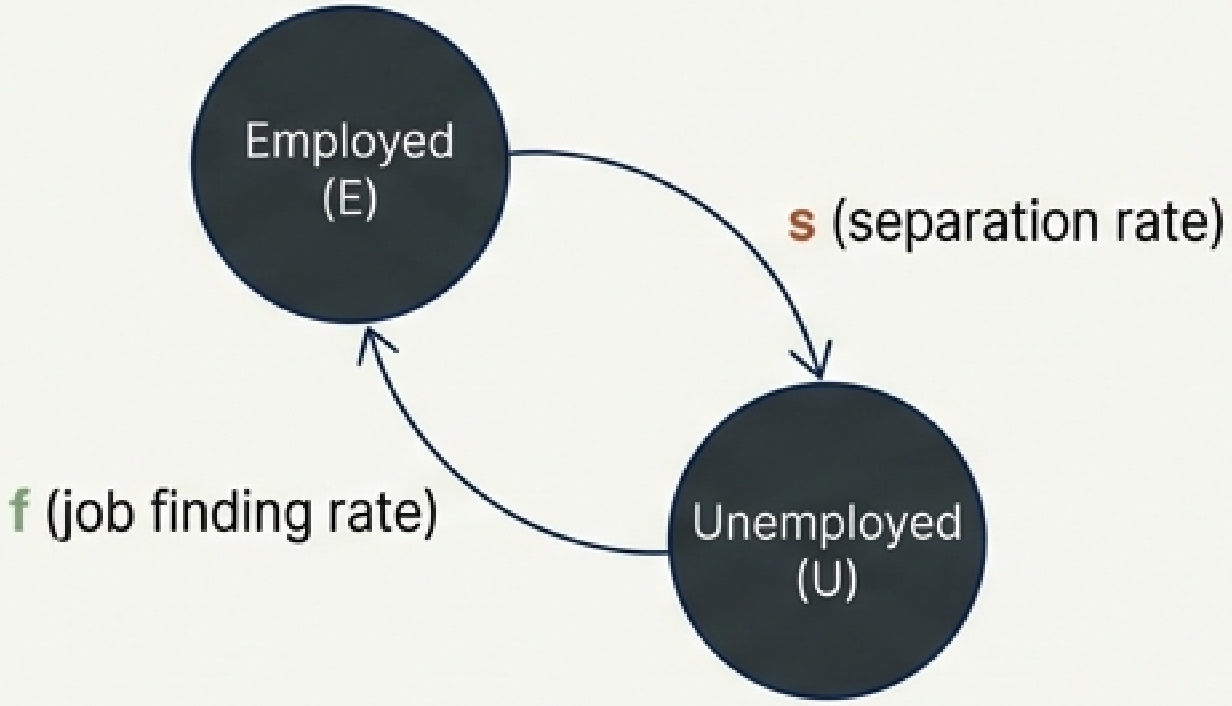
Modeling Growth, Risk, and Markets

1 The Tools (Stochastic Calculus & Risk Aversion)	**2 The Laboratory** (The Stochastic Growth Model)	**3 The Market** (Arrow Securities & Equivalence)	**4 The Reality** (Incomplete Markets & Inequality)

Quantifying Chaos: The Stochastic Process

Moving from deterministic sequences to random variables indexed by time.

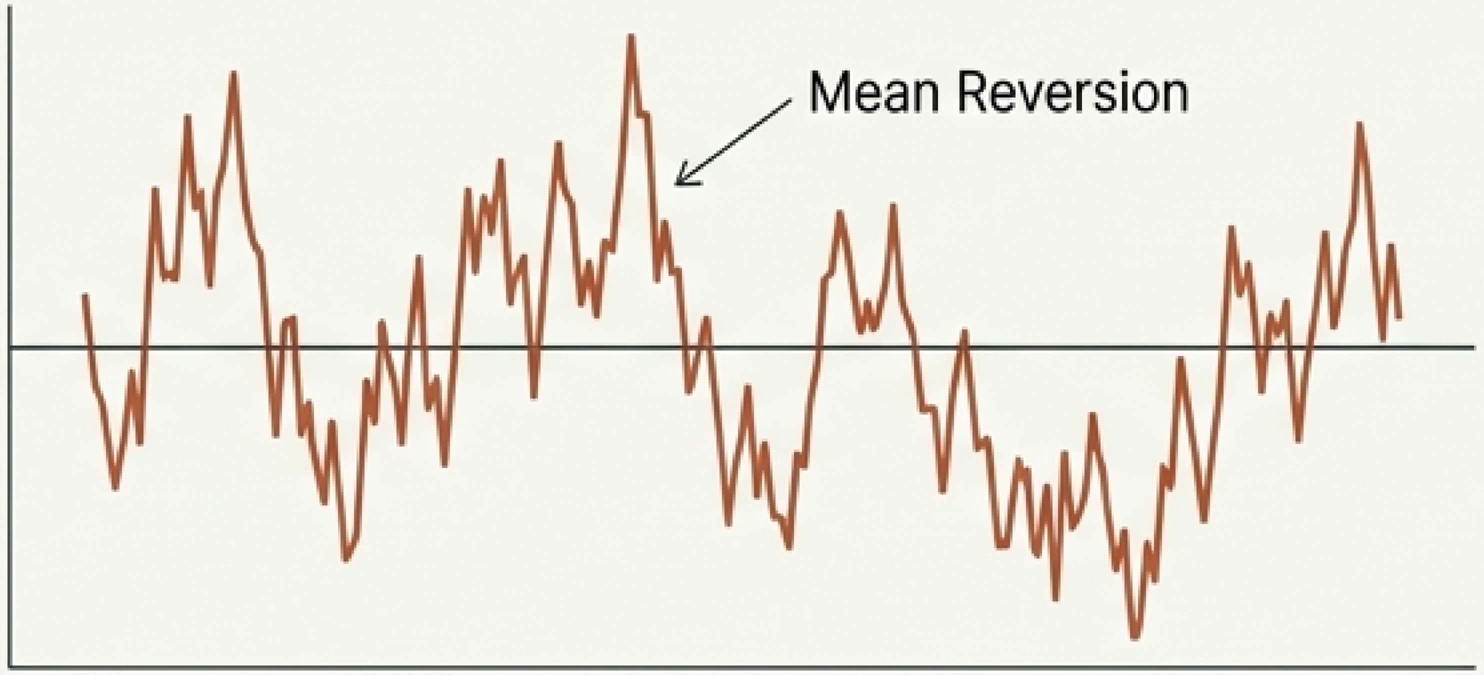
Discrete Time: The Markov Chain



$$P = \begin{bmatrix} (1-s) & s \\ f & (1-f) \end{bmatrix}$$

History doesn't matter. Only the current state predicts the future.

Continuous Time: The AR(1) Process



$$x_t = \rho x_{t-1} + b\varepsilon_t + (1-\rho)\mu$$

ρ (Persistence)　　b (Volatility)　　μ (Long-run Mean)

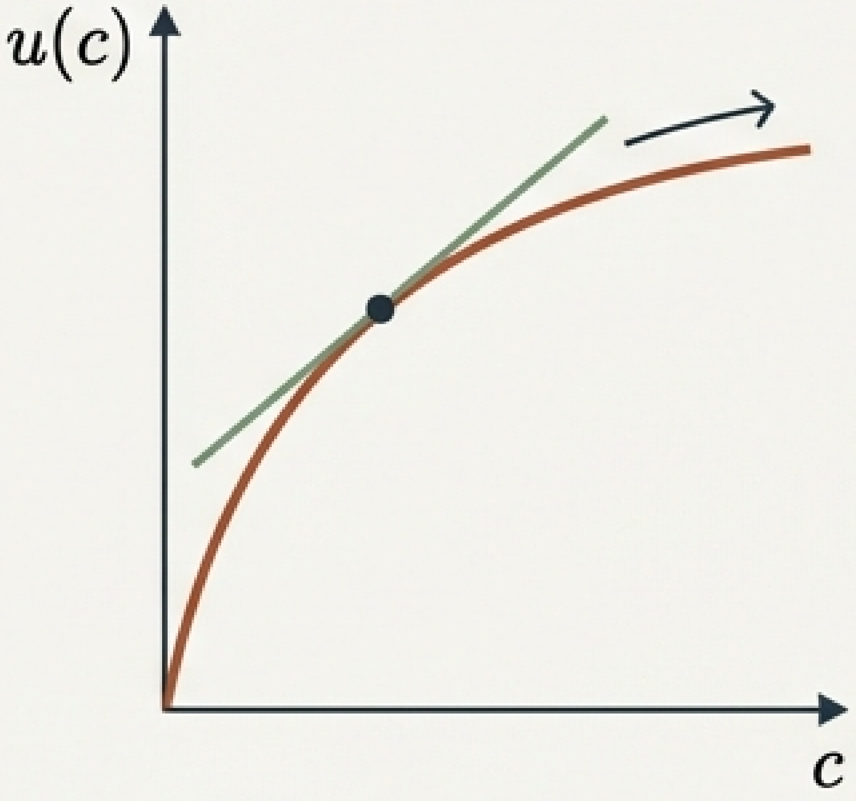
Key Concept: Stationarity. Shocks eventually fade; the variance does not grow infinitely.

The Human Element: Preferences

How agents rank risky consumption streams using Expected Utility Theory

The Core Assumption

Agents prefer smooth consumption paths. Utility functions are concave ($u'' < 0$).



Measuring Fear: Two Approaches

CARA (Constant Absolute Risk Aversion)

$$u(c) = -e^{-\alpha c}$$

Behavior: The rich and poor hold the same absolute dollar amount of risky assets.

Empirically Unrealistic

CRRA (Constant Relative Risk Aversion)

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}$$

Behavior: The rich and poor hold the same percentage of wealth in risky assets.

The Gold Standard for Growth Models

The Dual Role of σ

In power utility, the parameter σ controls two distinct forces:

1. Risk Aversion (Fear of uncertainty across states)
2. Intertemporal Substitution (Willingness to smooth across time)

The Planner's Problem

Optimization requires a "Contingent Plan"—deciding now for every future history.

The Patience: Discount factor.

The Expectation: Weighted average across all future shocks.

$$u'(C_t) = \beta E_t\left[\underline{u'(C_{t+1}) \cdot R_{t+1}}\right]$$

The Cost: The pain of saving today (marginal utility loss).

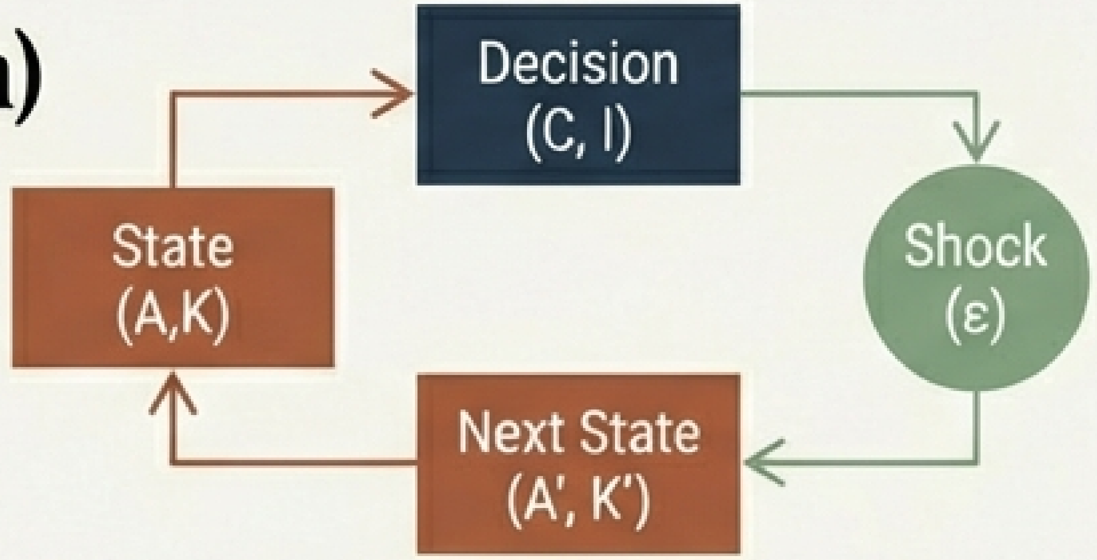
The Benefit: The return tomorrow, valued more highly in "bad states" (when consumption is low).

The Euler Equation Insight: The return to saving isn't just an average number. It is weighted by need. Returns that pay off during recessions (high marginal utility) are more valuable than returns during booms.

Cracking the Code: Recursion & Linearization

Transforming an infinite horizon problem into a solvable system.

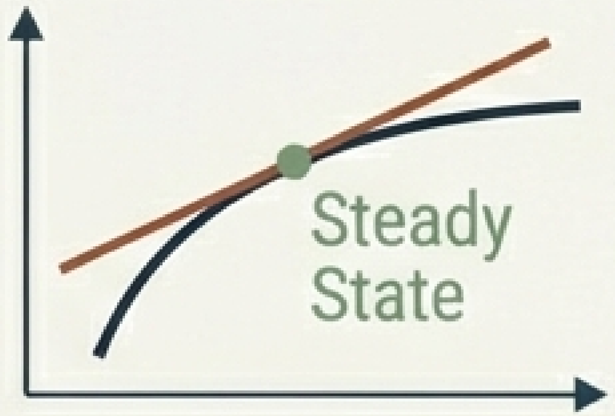
Step 1: The Recursive Formulation (Bellman Equation)



We replace infinite histories with State Variables (A, K).

$$V(A, K) = \max \{u(C) + \beta \sum \pi(A'|A)V(A', K')\}$$

Step 2: Linearization & Certainty Equivalence



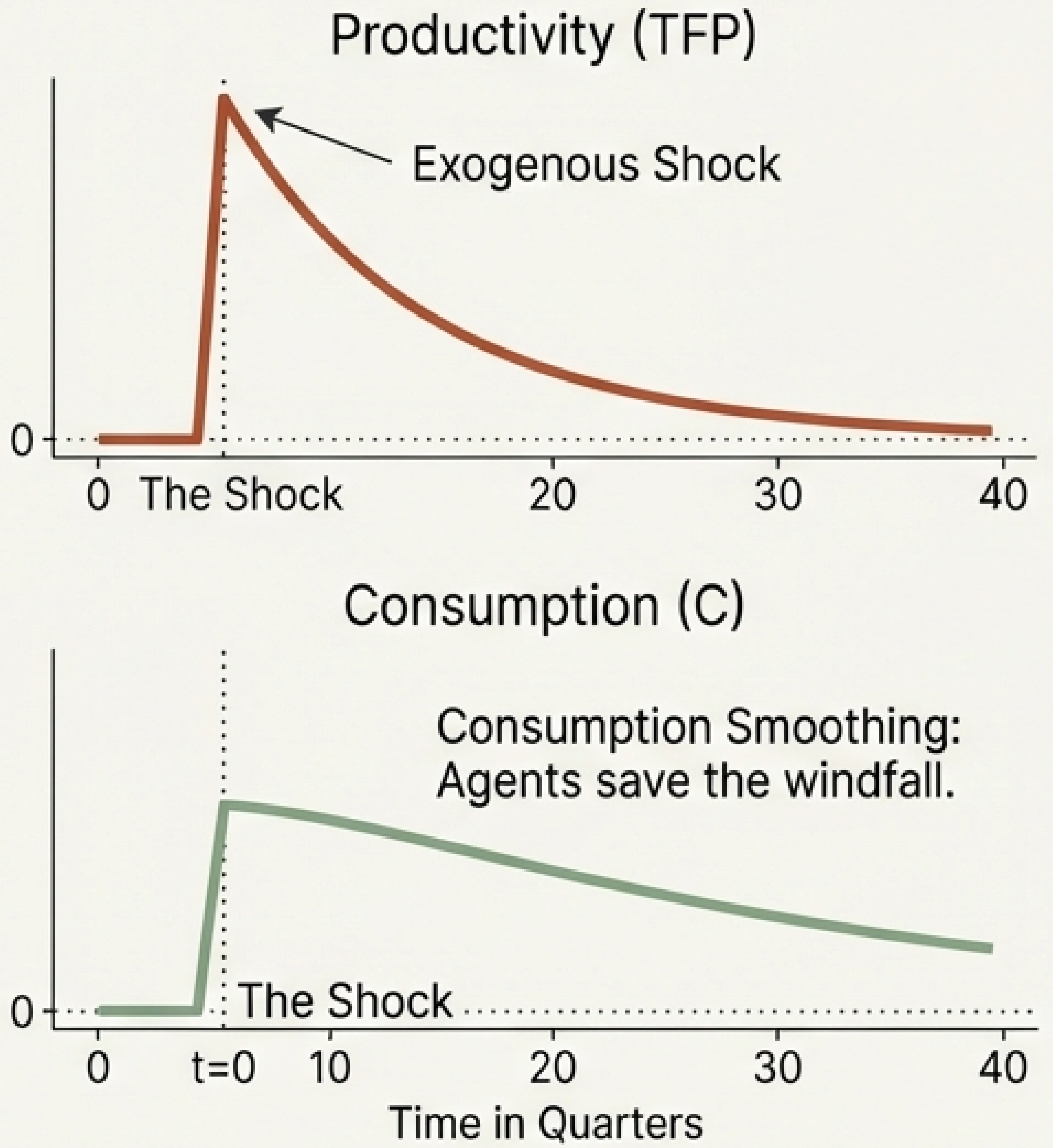
We approximate around the steady state. In a linear world, variance drops out.

$$K' = \bar{K} + g_K(K - \bar{K}) + g_A(A - \bar{A})$$

Agents behave as if the future shock will certainly be its expected value ($E[\epsilon'] = 0$).

The Ripple Effect: Impulse Responses

Visualizing how a TFP shock propagates through the economy.



Anatomy of a Shock:

1. TFP jumps.
2. Output rises.
3. Agents do not consume the entire surplus; they save (Capital rises).
4. This spreads the benefit of the shock over many years.

The Market Solution: Arrow-Debreu Securities

Trading risk by buying insurance for specific future histories.

The Pricing Mechanism

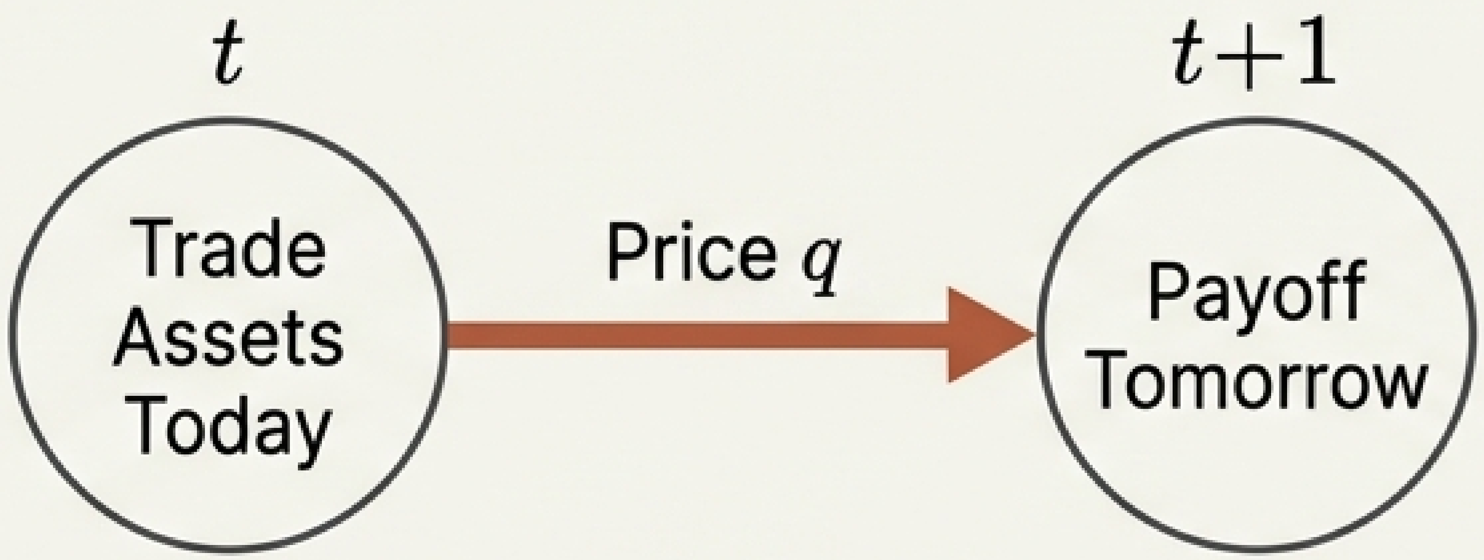
1. **Probability:** How likely is this event? ($\pi(\omega_t)$)
2. **Scarcity:** Is the economy poor in this state? (Marginal Utility)

$$p_t(\omega_t) = \beta^t \pi(\omega_t) \frac{u'(C_t)}{u'(C_0)}$$

"If markets are complete, agents can trade away all **idiosyncratic risk.** Consumption becomes **perfectly smooth relative to individual income.**"

Sequential Trading

Replicating complete markets through repeated trading over time.



The No-Ponzi Condition

$$a_t \geq \text{Natural Borrowing Limit}$$

You cannot roll over debt forever. Eventually, you must be able to repay using future income.

The Budget Constraint

$$c_t + \sum q \cdot a_{t+1} = y_t + a_t$$

Instead of one massive trade at the beginning of time, agents trade one-period assets repeatedly.

Key Result

If there are as many assets as there are future states ($\mathrm{Rank}(D) = S$), **sequential trading is equivalent to Arrow-Debreu.**

The Equivalence Result

The First Welfare Theorem holds even in a stochastic world.

The Social Planner

- Goal: Maximize Welfare.
- Constraint: Physical Resources.

FOC: $u'(C) = \beta E\left[u'(C')f_k(\cdot)\right]$

$=$

The Competitive Market

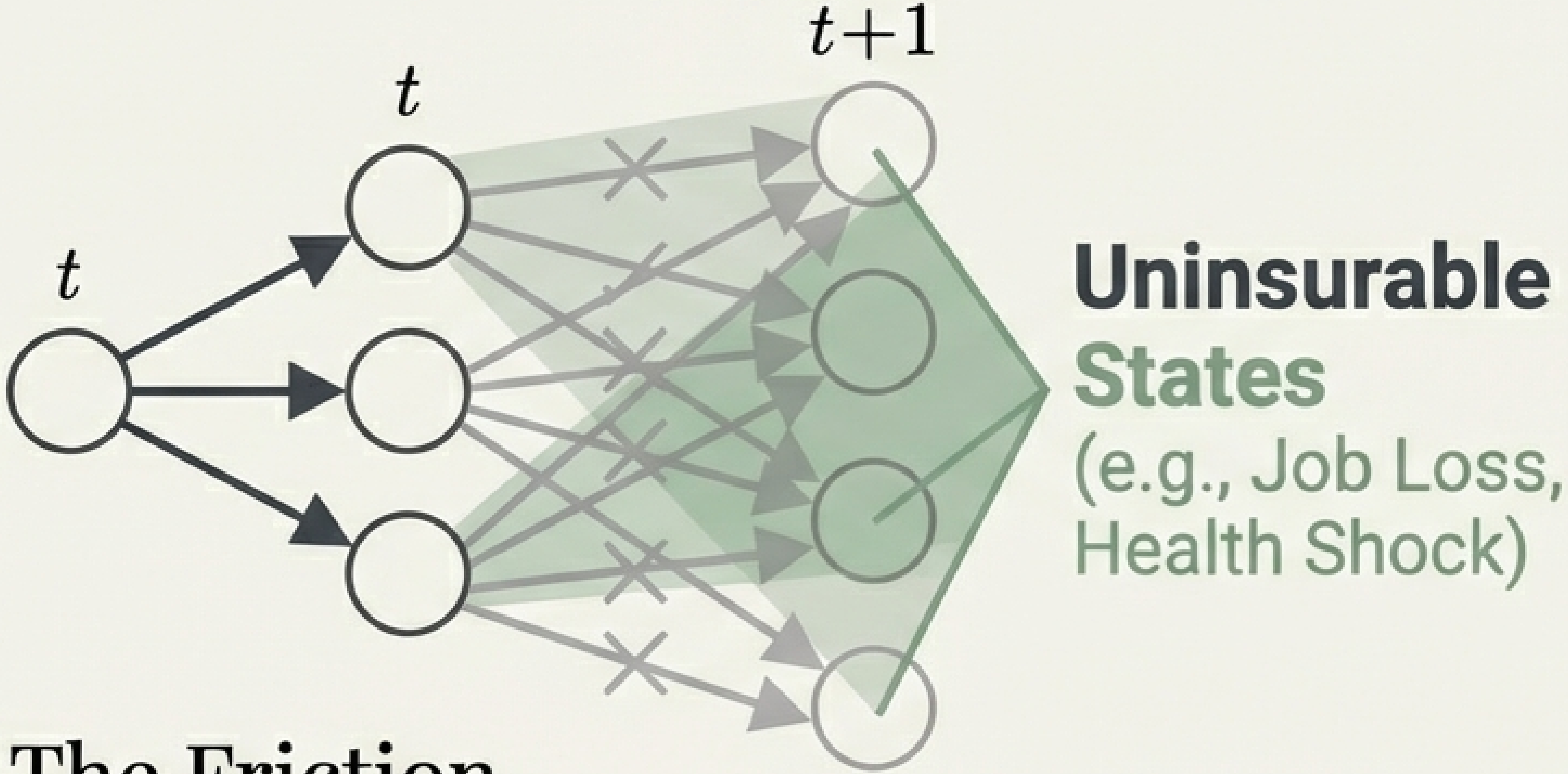
- Goal: Maximize Utility / Profit.
- Constraint: Budget / Prices.

FOC: $u'(c) = \beta E\left[u'(c')(r + 1 - \delta)\right]$

Since the firm sets $r = f_k - \delta$, the equations are identical. The Invisible Hand achieves the same optimal allocation as the Benevolent Dictator.

The Reality Check: Incomplete Markets

When the armor has cracks: Uninsurable Idiosyncratic Risk.



The Friction

In reality, we cannot buy insurance for every specific event. We can often only trade a **risk-free bond**.

Equation Comparison

- Complete Markets: c_i is constant (full insurance).
- Incomplete Markets:

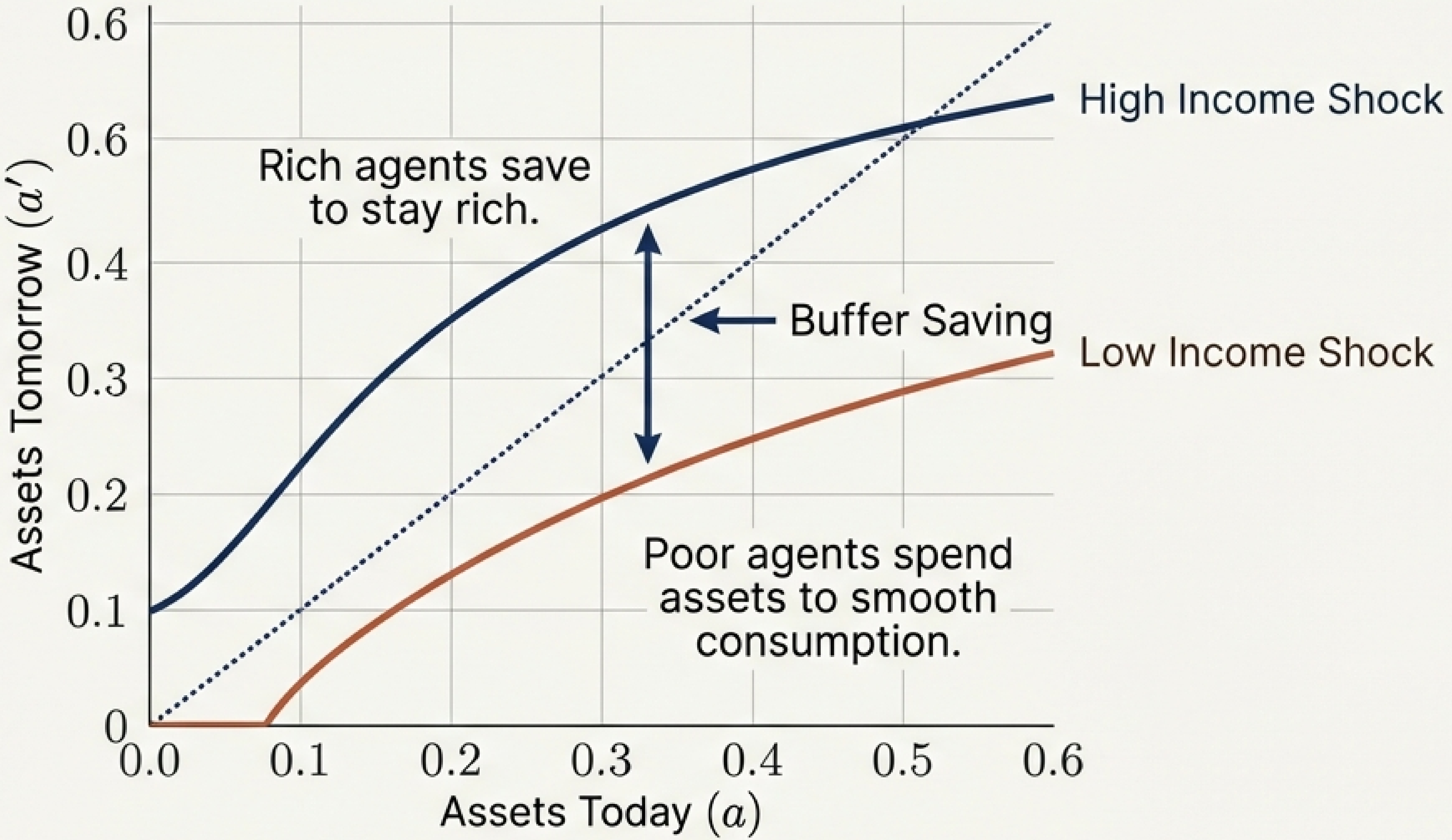
$$u'(c_i) = (1 + r)\beta E[u'(c_i')]$$

Consumption **MUST** fluctuate with individual income history.

Takeaway: Agents are no longer identical. Inequality emerges.

Self-Insurance: The Buffer Stock

How agents use savings to survive uninsurable risk.



Agents accept some volatility to avoid the infinite cost of perfect safety.

The Stationary Distribution

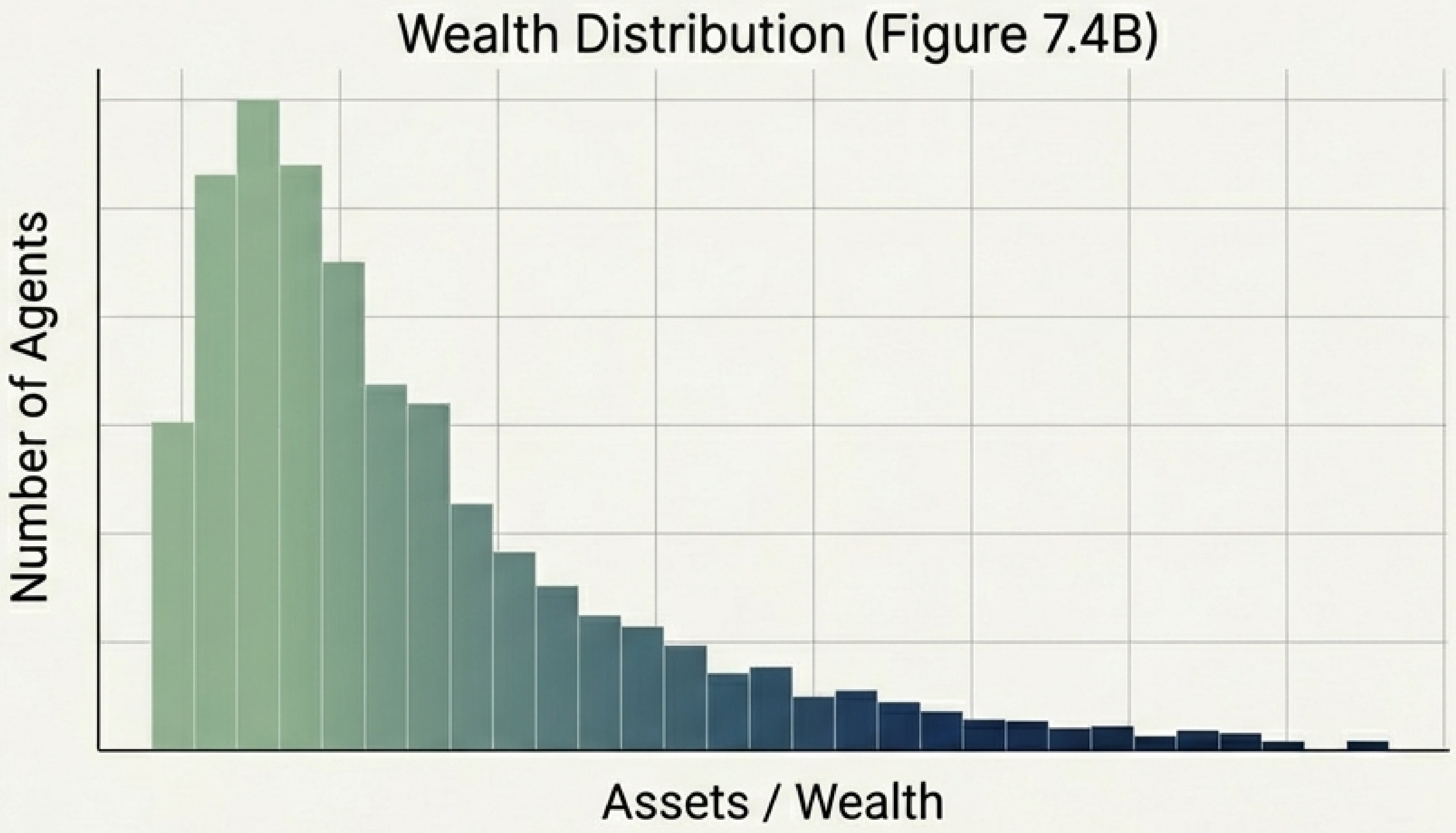
Inequality as an Equilibrium Outcome.

If we simulate thousands of agents facing incomplete markets:

- Some get lucky (sequence of high income).

- Some get unlucky (sequence of low income).

The result is not a single steady-state number, but a permanent distribution of inequality.



Key Insight: Wealth distribution is endogenous to the history of shocks.

Synthesis: Taming Uncertainty

From Chaos to Distributions

The Beast

Stochastic Shocks (TFP)

→

The Defense

Risk Aversion & Smoothing

→

The Market

Equivalence Result (Complete)

→

The Reality

Incomplete Markets & Inequality

Uncertainty is not just noise to be averaged out. It drives the fundamental structure of asset prices, savings behavior, and the distribution of wealth.