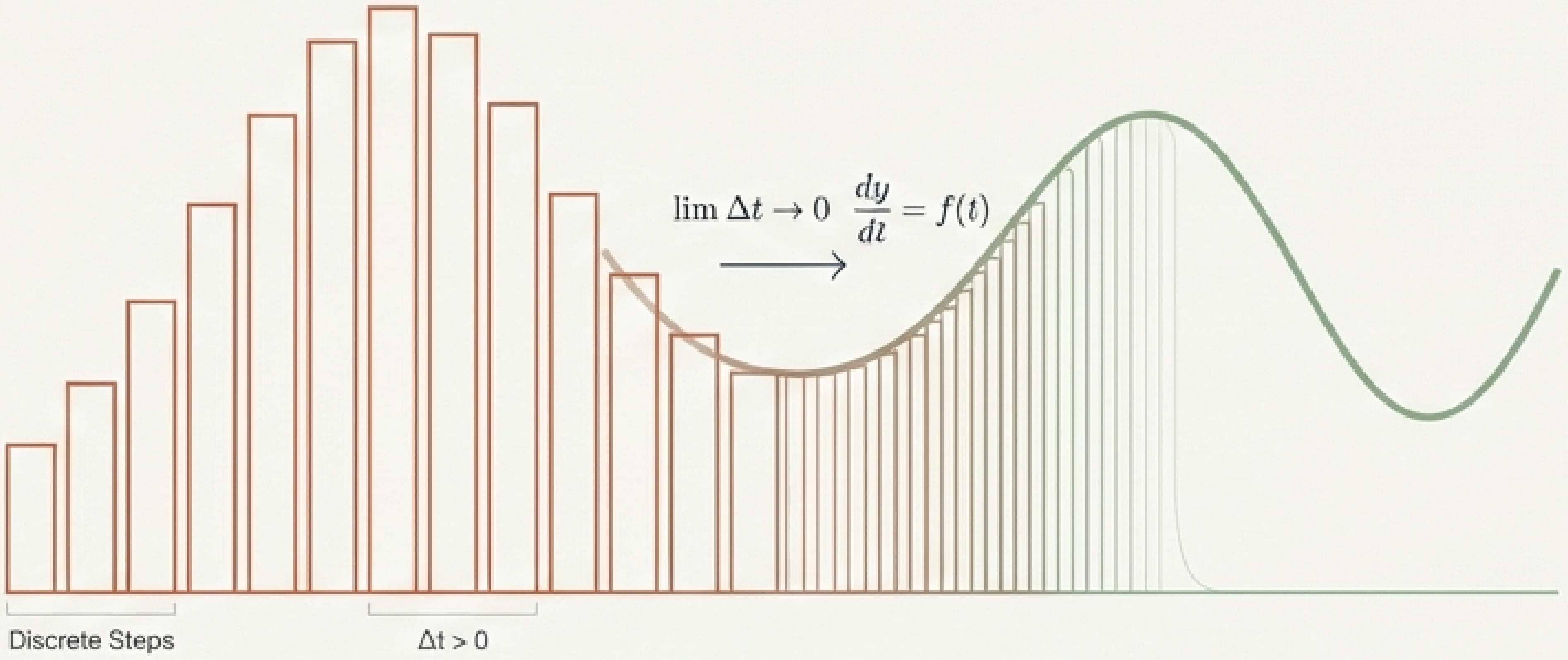
Continuous-Time Analytical Techniques

Modeling Smooth Economic Dynamics & Optimization



A Reference Guide to Chapter 9 Concepts

Bridging Discrete Measurement and Continuous Theory

Discrete Time

The Language of Data

Economic data arrives in steps (monthly, quarterly, yearly). Discrete time aligns with empirical measurement and numerical simulation.

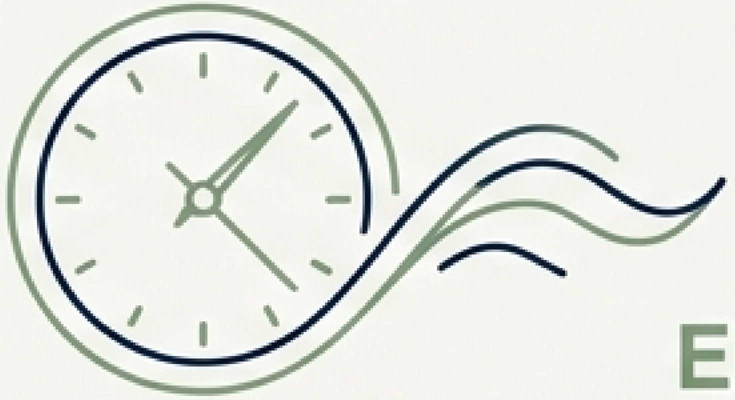
$$X_{t+1} - X_t$$

Intuitive & Measurable

Continuous Time

The Language of Theory

Economic reality is a constant flow. Continuous time allows for analytical tractability, closed-form solutions, and transparent logic.



$$\frac{dX(t)}{dt}$$

Elegant & Tractable

Summary: While data is discrete, the underlying economic forces are continuous flows. We use continuous models to capture the elegance of these flows.

The Rosetta Stone: Translating the Syntax of Time

	Concept	Discrete Notation (Steps)	Continuous Notation (Flows)
Variable Representation	Time Indexing	X_t (Sequence of values)	$X(t)$ (Function of time, $t \in \mathbb{R}_+$)
Rate of Change	Change over Time	$\Delta X_t = X_{t+1} - X_t$	$\dot{X}(t) = \frac{dX(t)}{dt}$ (Newton's Dot Notation)
Growth Dynamics	Constant Growth	$X_t = (1+\gamma)^t X_0$	$X(t) = e^{\gamma t} X(0)$

Key Insight: Differential equations generate continuous functions, not just sequences.

NotebookLM

Growth Dynamics & The Power of Logs

Converting multiplicative functions into additive growth equations.

The Mathematical Rule

$$\frac{\dot{X}(t)}{X(t)} = \frac{d}{dt}\ln(X(t))$$

Application to Production

Cobb-Douglas Production Function

$$Y(t) = z(t)K(t)^{\alpha}L(t)^{1-\alpha}$$

Take Natural Logs

Log-Linear Form

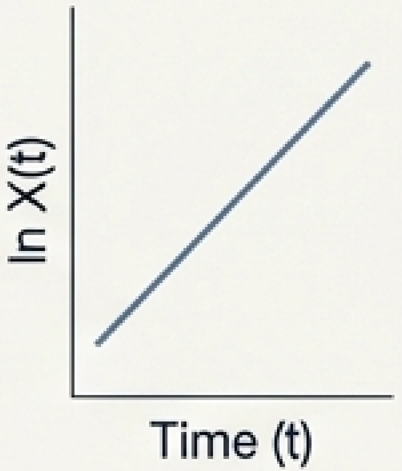
$$\ln Y(t) = \ln z(t) + \alpha \ln K(t) + (1-\alpha)\ln L(t)$$

Differentiate w.r.t Time

Growth Accounting Equation

$$\frac{\dot{Y}(t)}{Y(t)} = \frac{\dot{z}(t)}{z(t)} + \alpha\frac{\dot{K}(t)}{K(t)} + (1-\alpha)\frac{\dot{L}(t)}{L(t)}$$

Constant growth becomes linear in logs.



Intertemporal Optimization and Discounting

The Discrete Objective

Maximizing the sum of discounted utility.

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

β: Discount Factor (Weight on future)

The Continuous Objective

Maximizing the integral of discounted flow utility.

$$\max \int_0^{\infty} e^{-\rho t} u(c(t)) dt$$

ρ: Discount Rate (Impatience parameter)

Transformation

The Limit Concept: From β to ρ

1. Discrete discount factor over period Δ:
2. Discount over time t (t/Δ periods):
3. Limit as $\Delta \to 0$:

$$\beta \approx \frac{1}{1+\rho\Delta} \longrightarrow \left(\frac{1}{1+\rho\Delta}\right)^{t/\Delta} \longrightarrow e^{-\rho t}$$

The Maximum Principle: The Hamiltonian

The continuous-time engine for dynamic optimization.

$$H(t) \equiv \underbrace{e^{-\rho t}u(c(t))}_{\text{Current Utility}} + \underbrace{\mu(t)}_{\text{Shadow Price}}\underbrace{(ra(t) + w - c(t))}_{\text{Change in State}}$$

Control Variable ($c(t)$): Consumption. The choice made at instant t to maximize H.

Costate Variable ($\mu(t)$): The Lagrange multiplier. Represents the marginal value (shadow price) of increasing the state variable.

State Variable ($a(t)$): Assets. The variable inherited from the past that constrains the present.

The Hamiltonian captures the total value of current decision-making: the direct utility plus the value of the assets accumulated for the future.

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The Optimization Recipe: First-Order Conditions (FOCs)

Step 1: Optimality Condition (Control)

Maximize the Hamiltonian with respect to the control variable.

$$\frac{\partial H(t)}{\partial c(t)} = 0 \implies e^{-\rho t} u'(c(t)) = \mu(t)$$

Marginal Utility = Shadow Price

Step 2: Multiplier Equation (Costate)

Describe the evolution of the shadow price.

$$\frac{\partial H(t)}{\partial a(t)} + \dot{\mu}(t) = 0 \implies \dot{\mu}(t) = -r\mu(t)$$

Arbitrage condition for the value of assets.

Step 3: Transversality Condition (TVC)

Boundary condition at infinity (No value left on the table).

$$\lim_{T \to \infty} e^{-\rho T} u'(c(T)) a(T) = 0$$

Different from Discrete Time: In continuous time, we evaluate $\frac{\partial H(t)}{\partial a(t)}$, $\partial a(t)$, not the next period's state.

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The Continuous Euler Equation

The fundamental rule for consumption smoothing.

$$\frac{\dot{c}(t)}{c(t)} = \frac{r - \rho}{\sigma}$$

- r: Interest Rate (Reward for saving)
- ρ: Discount Rate (Impatience)
- σ: Coefficient of Relative Risk Aversion (Resistance to fluctuation)

Intuition Box

Interpreting the Trade-off:

- **If $r > \rho$:** The market reward exceeds impatience. Agents save, and consumption *grows* over time ($\dot{c} > 0$).
- **If $r < \rho$:** Impatience dominates. Agents borrow/dissave, and consumption *falls* ($\dot{c} < 0$).
- **Role of σ:** Determines the speed of adjustment. High risk aversion (σ) implies slower changes in consumption for the same interest rate gap.

Discrete Parallel: $\frac{u'(c_t)}{u'(c_{t+1})} = \beta(1 + r)$

Application I: The Solow Growth Model

Translating the Law of Motion to Continuous Time

Model Inputs

Production

$$Y(t) = F(K(t), A(t)L(t))$$

Capital Accumulation

$$\dot{K}(t) = sY(t) - \delta K(t)$$

Exogenous Growth

Labor $\frac{\dot{L}}{L} = n$ Technology $\frac{\dot{A}}{A} = \gamma$

Derivation (Intensive Form)

Define capital per effective worker $\tilde{k} \equiv K/(AL)$. Differentiating with respect to time:

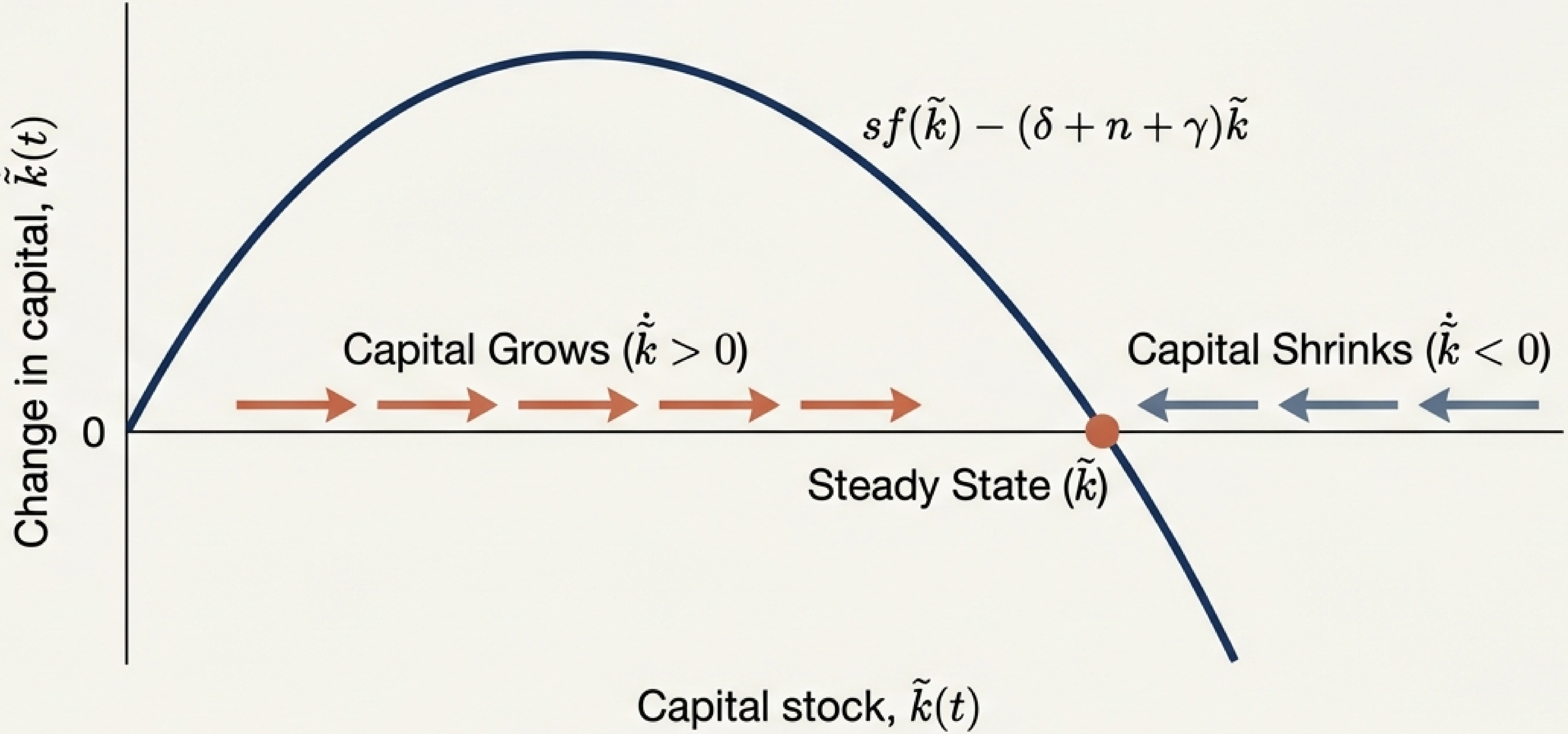
$$\frac{\dot{\tilde{k}}}{\tilde{k}} = \frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L}$$

The Fundamental Differential Equation

$$\dot{\tilde{k}}(t) = \underbrace{sf\left(\tilde{k}(t)\right)}_{\text{Actual Investment}} - \underbrace{(\delta + n + \gamma)\tilde{k}(t)}_{\text{Break-even Investment}}$$

Change in Capital = [Actual Investment] minus [Break-even Investment]

Visualizing Solow Dynamics



Stability: Regardless of the starting point, the economy slides naturally toward the steady state where investment equals break-even requirements.

Application II: The Ramsey Model

From fixed savings rates to optimal intertemporal choice.

The Social Planner's Problem

Maximize Welfare:

$$\max \int_0^\infty e^{-\rho t} \frac{c(t)^{1-\sigma}}{1-\sigma} dt$$

The Dynamic System

1. Resource Constraint (Capital Motion):

$$\left[\ \dot{k}(t) = f(k(t)) - \delta k(t) - c(t)\ \right].$$

(Describes the physical limits of the economy.)

2. Euler Equation (Consumption Motion):

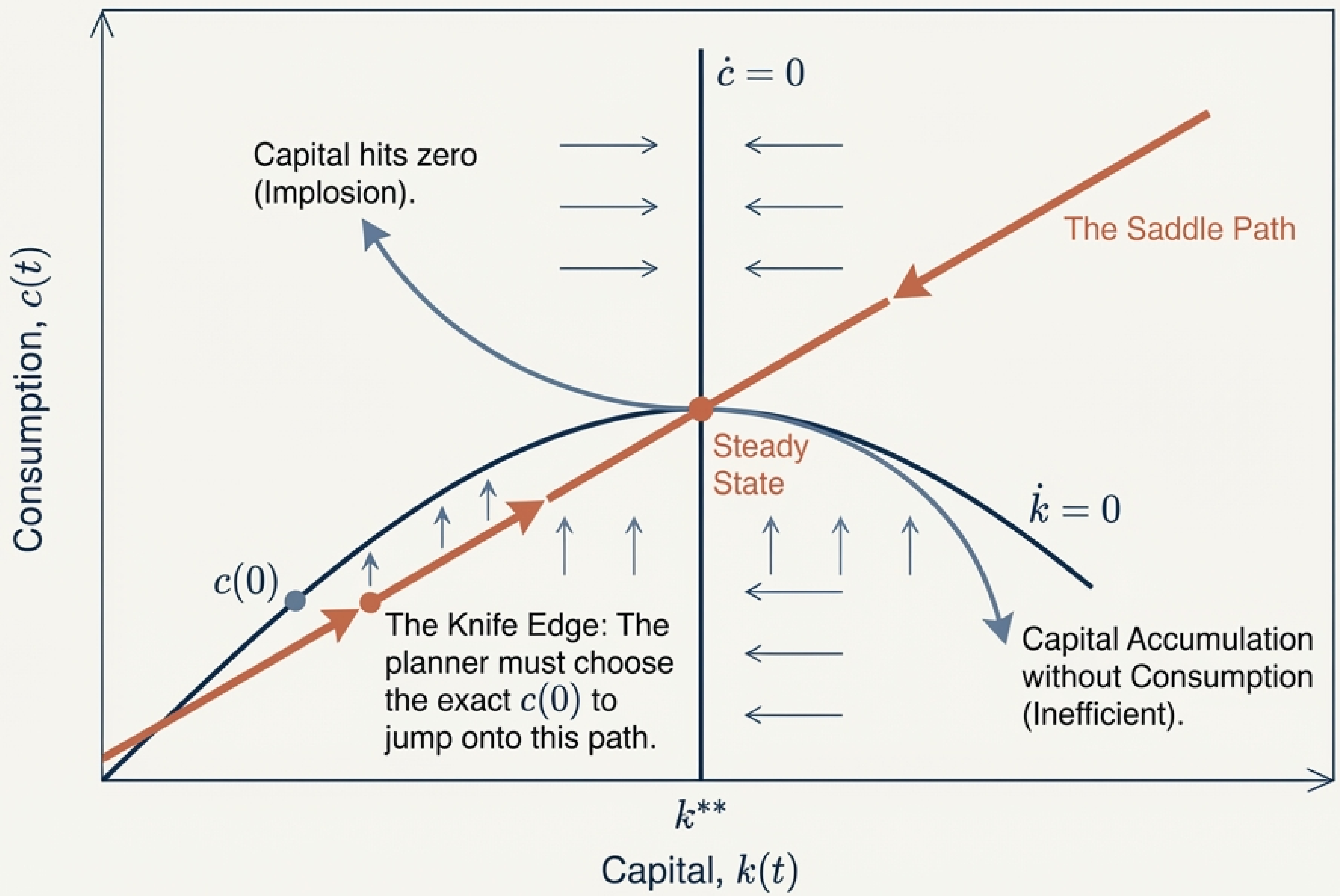
$$\left[\ \frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}\left[f'(k(t)) - (\delta + \rho)\right].\ \right]$$

(Describes the optimal psychological choice.)

Steady State Condition: The Modified Golden Rule (k^{**})

$$f'(k^{**}) = \delta + \rho$$

The Geometry of Optimal Growth: The Saddle Path



Decentralized Market Equilibrium

Does the invisible hand match the planner's solution?

Households

Maximize Utility.
Supply Labor/Capital.

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}(r - \delta - \rho)$$

$\neq$

Market Prices

Firms pay marginal products.

$$r(t) = f'(k(t))$$
$$w(t) = f(k) - kf'(k)$$

$\neq$

Result

Substituting prices into household decisions yields:

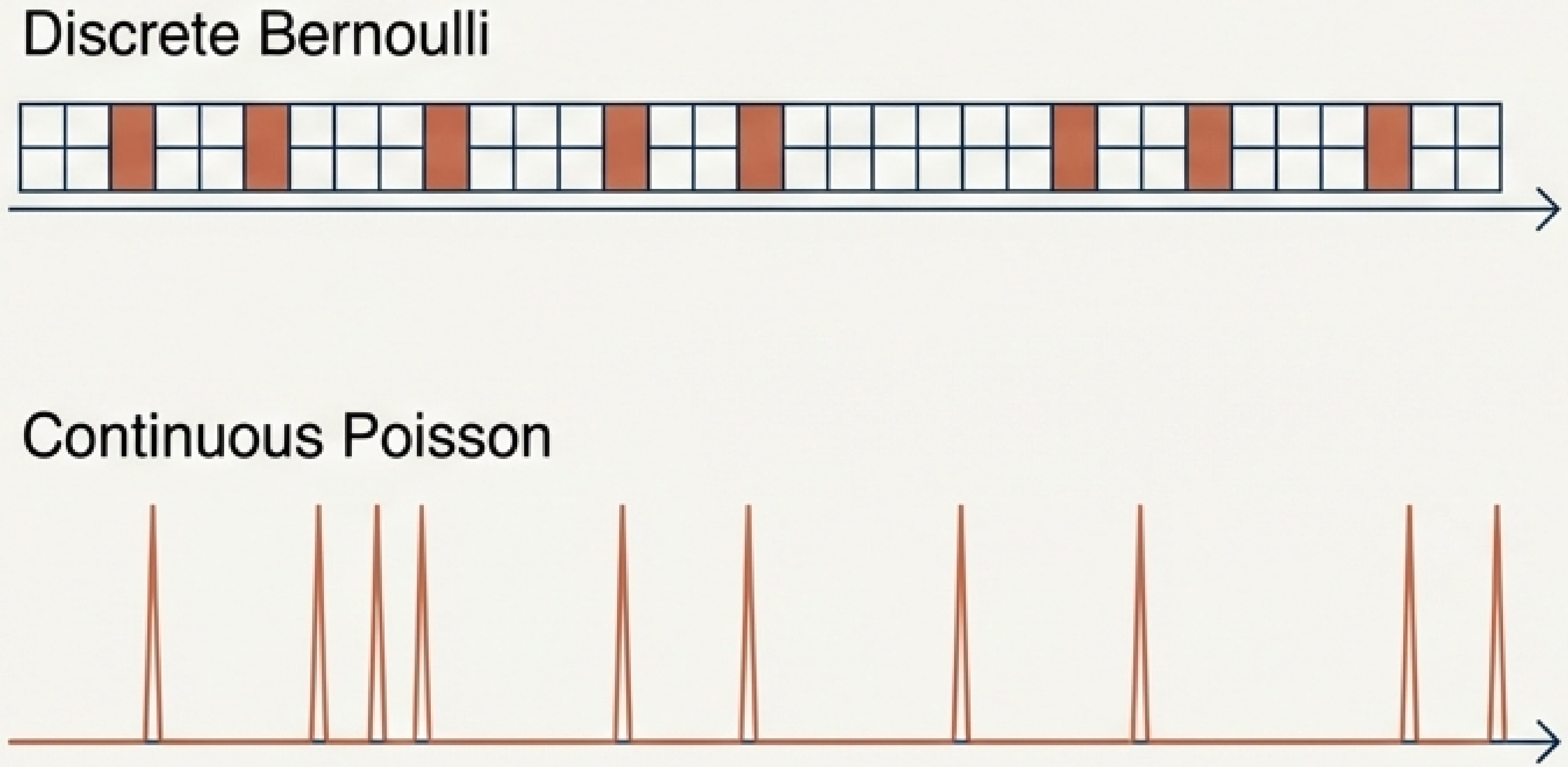
$$\frac{\dot{c}}{c} = \frac{1}{\sigma}(f'(k) - \delta - \rho)$$

Pareto Optimality

The Decentralized Market Equilibrium generates the exact same differential equations as the Social Planner. The **First Welfare Theorem** holds.

Modeling Uncertainty: The Poisson Process

Handling shocks in continuous time without discrete periods.



Transition from Binomial to Poisson:

1. Slice time into n sub-periods.
2. Probability of shock scales with time: λ/n.
3. Limit as $n \to \infty$:

Probability of k events:

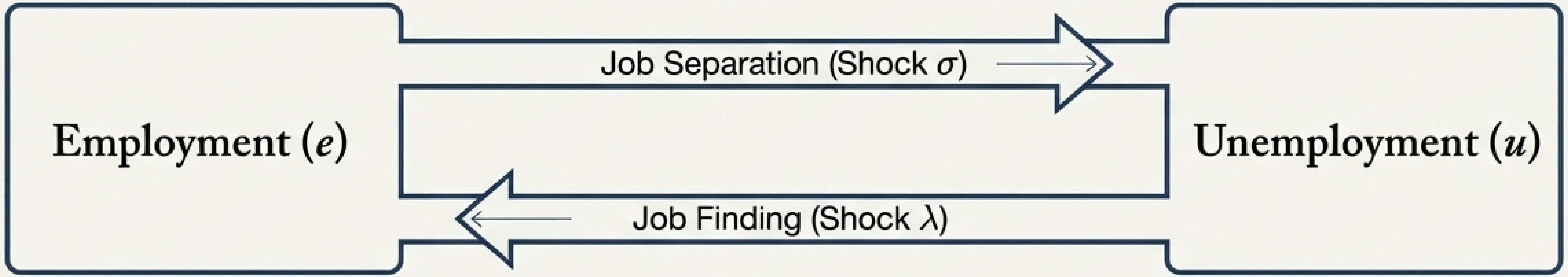
$$p(k) = \frac{e^{-\lambda T}(\lambda T)^k}{k!}$$

Key Properties List:

- **Memoryless**: Past duration does not influence future probability.
- **Survival Probability**: Chance of no event by time t is $e^{-\lambda t}$.

Example: Unemployment Dynamics

Random individual shocks creating smooth aggregate flows.



$$\dot{u}(t) = \underbrace{(N - u(t))\sigma}_{\text{Inflows}} - \underbrace{u(t)\lambda}_{\text{Outflows}}$$

Set $\dot{u} = 0$ to find the Natural Rate of Unemployment:

$$\bar{u} = \frac{\sigma N}{\sigma + \lambda}$$

Conclusion: Even with stochastic Poisson shocks at the individual level, the aggregate macro system evolves according to a deterministic differential equation.