Chapter 3: The Solow Model

Graduate Macroeconomics Slides

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Outline

Motivation

- Prevous chapter showed that $\frac{K_t}{Y_t}$ is nearly constant over time
- Earlier theories (e.g., Harrod–Domar) treated this stability as a technological property.
- Solow's approach: Capital and labor are (very) substitutable in production, yet $\frac{K_t}{Y_t}$ remains stable.
- Led to a new framework for understanding:
  - Macroeconomic dynamics
  - Measurement of technological change

Chapter Goals

- Introduce the **Solow model**: (Solow, 1956)
  - Basic assumptions
  - Aggregate production function
  - Capital accumulation and steady state
- Extensions to include:
  - Technological progress
  - Population growth
  - Balanced growth path
- Show how the model explains:
  - Long-run per-capita income growth
  - Convergence across economies

Aggregate Production Function

Setup:

$$Y_t = F(K_t, L_t).$$

Assumptions:

1. Strictly increasing in both K and L.
2. Strictly quasiconcave (convex isoquants).
3. Constant returns to scale.
4. $F(0, L) = 0$.
5. Inada conditions: $\lim_{K\to 0} F_K(K, L) = \infty$ and $\lim_{K\to\infty} F_K(K, L) = 0$.

Per-Capita Notation

- Assume labor L_t constant initially; normalize to 1.
- Then $y_t = f(k_t)$ with $f(k) = F(k, 1)$.
- Capital accumulation:

$$k_{t+1} = i_t + (1 - \delta)k_t,$$

  where $\delta \in (0, 1)$ is depreciation.
- Goods market equilibrium:

$$y_t = c_t + i_t.$$

- Constant saving rate s: $i_t = sy_t$.

$$\implies k_{t+1} = sf(k_t) + (1 - \delta)k_t.$$ (**Fundamental equation of the Solow model.**)

Steady State

Steady state: $k_{t+1} = k_t = \bar{k}$.
Then,

$$\bar{k} = s\, f(\bar{k}) + (1-\delta)\,\bar{k} \quad \Longrightarrow \quad \delta\,\bar{k} = s\, f(\bar{k}).$$

Existence and uniqueness:

- By Inada conditions and concavity, there is a unique $\bar{k} > 0$ solving this.

Dynamics: If $k_0 < \bar{k}$, then k_t grows monotonically toward $\bar{k}$. If $k_0 > \bar{k}$, then k_t falls until reaching $\bar{k}$.

Cobb–Douglas Example

$$Y_t = K_t^{\alpha} L_t^{1-\alpha},$$

then per-capita form $f(k) = k^{\alpha}$. Steady-state capital:

$$\bar{k} = \left(\frac{s}{\delta}\right)^{\frac{1}{1-\alpha}},$$

and hence steady-state output per capita $\bar{y} = \bar{k}^{\alpha}$.

- $\bar{k}$ increases in s and decreases in δ.
- $\bar{k}/\bar{y}$ is constant.

Dynamics in the Solow Model

Figure: Dynamics in the Solow model

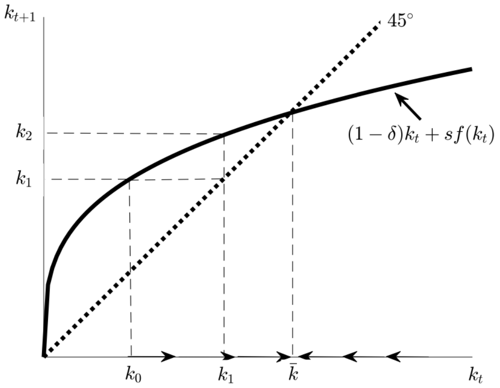

Figure 3.1: Dynamics in the Solow model.

- Figure 1 is a diagram to analyze the dynamics of k_t when $k_0 > 0$ is not at the steady- state level.
- It plots the Solow equation with the 45-degree line (that is, representing $k_{t+1} = k_t$) In the figure, the intersection of Solow equation and the 45-degree line represents the steady-state $\bar{k}$.

Other Kinds of Dynamics

- The growth model can, in principle, generate very rich and complex dynamics if its neo-classical feature is not present, (i.e., if the production function is not strictly concave in capital.).

Endogenous growth

- Growth driven internally (without exogenous technological progress) when capital's marginal product $F_1(k, 1)$ exceeds δ/s.
- $F_1(k, 1) > \delta/s \Rightarrow$ No steady state exists $\Rightarrow k_t$ grows indefinitely.

Example: Linear Production: $y_t = Ak_t$ (ignores labor, $\alpha = 1$)

- Identical countries with different initial k_0 remain divergent (gap persists).

Limitations: Empirically implausible: Labor accounts for $\sim$2/3 of income.

- Decreasing returns to capital make unbounded growth unlikely in practice.

Endogenous Growth in the Solow model

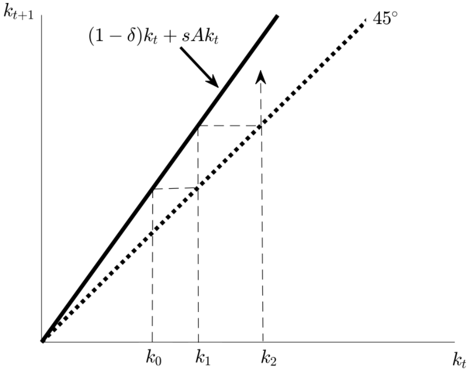

Figure 3.2: Endogenous growth in the Solow model.

Figure: Endogenous Growth in the Solow model

Poverty Traps in the Solow model

- A poverty trap occurs when an economy gets stuck in a low steady state ($\bar{k_1}$) due to multiple steady states in the Solow model.
- **Non-concave production function:** Increasing returns in some regions (e.g., infrastructure investments) lead to multiple intersections with the 45-degree line.
- ($\bar{k_1}$)(low) and ($\bar{k_3}$)(high) are stable.
- ($\bar{k_2}$) is unstable (small perturbations push the economy away.)

- Economies with low initial capital ($\bar{k}_0$) converge to $\bar{k}_1$ and remain trapped.
- **Escape** requires temporarily increasing savings or capital to push the economy past $\bar{k}_2$

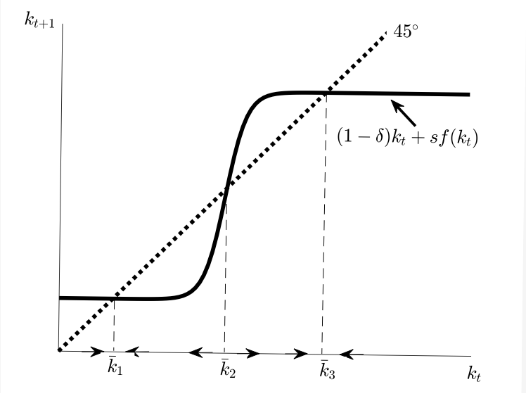

Figure 3.3: Poverty traps in the Solow model

Figure: Poverty Traps in the Solow model

Non-Monotonic Dynamics and Chaos

- If the production function $f(k)$ declines steeply at high capital levels, the Solow model can exhibit oscillatory or chaotic dynamics.
- think of **Bakery analogy:** Adding too many ovens reduces productivity due to overcrowding.
- Capital k_t may oscillate forever or exhibit chaotic behavior, never settling into a steady state.
- **Chaos:** Extreme sensitivity to initial conditions; no repeating patterns.

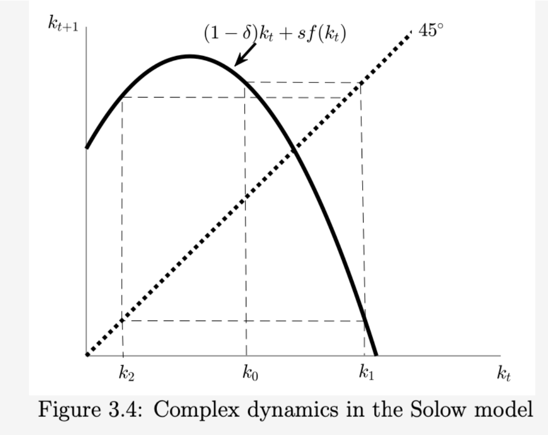

Figure: Complex Dynamics in the Solow model

Adding Population and Tech Growth

Extended model:

$$Y_t = F(K_t, A_t L_t),$$

where

$$L_t = L_0 (1+n)^t, \quad A_t = A_0 (1+\gamma)^t.$$

- A_t = labor-augmenting technology.
- n = population growth; γ = tech growth.

Effective-Labor Notation

Define

$$\tilde{k}_t = \frac{K_t}{A_t L_t}.$$

Rewrite accumulation:

$$K_{t+1} = s\,F(K_t, A_t L_t) + (1-\delta)\,K_t,$$

divide both sides by $A_{t+1}L_{t+1}$ gives:

$$\tilde{k}_{t+1} = \frac{s\,f(\tilde{k}_t)}{(1+\gamma)(1+n)} + \frac{1-\delta}{(1+\gamma)(1+n)}\,\tilde{k}_t.$$

$\Longrightarrow$ **Fundamental equation in effective-labor form.**

Balanced Growth Path (BGP)

- Along the BGP: $\tilde{k}_t = \bar{\tilde{k}}$ (constant).
- Solve:

$$(1+\gamma)(1+n)\,\bar{\tilde{k}} \;=\; s\,f(\bar{\tilde{k}}) + (1-\delta)\,\bar{\tilde{k}}.$$

- If f is Cobb–Douglas,

$$\bar{\tilde{k}} = \left(\frac{s}{(1+\gamma)(1+n)+\delta-1}\right)^{\frac{1}{1-\alpha}}.$$

- On the BGP, capital and output in *levels* keep growing at rates $n+\gamma$.

Long-Run Growth in y_t

Per-capita output:

$$y_t = \frac{Y_t}{L_t} = \frac{F(K_t, A_t L_t)}{L_t} = A_t f(\tilde{k}_t).$$

- If $\tilde{k}_t \to \bar{\tilde{k}}$, then $y_t \approx A_t f(\bar{\tilde{k}})$.
- Growth rate of y_t is γ in the long run.
- $\Longrightarrow$ **Key Solow result:** γ (tech growth) drives long-run per-capita growth. Changing s or δ only shifts levels, *not* the ultimate growth rate.

Stylized Facts Revisited

1. $\frac{K_t}{Y_t}$ is roughly constant in the long run. $\rightarrow$ In Solow, on the BGP, $\frac{K_t}{Y_t}$ is indeed constant.
2. $y_t = \frac{Y_t}{L_t}$ grows $\approx \gamma$. $\rightarrow$ Exactly matches Solow's BGP result.
3. Factor shares are stable. $\rightarrow$ Under competitive markets, α and $1 - \alpha$ remain constant in Cobb–Douglas.
4. Return to capital is stable. $\rightarrow$ $r_t = \partial F / \partial K$ depends on $\tilde{k}_t$. On BGP, it is constant.

Convergence

Solow's convergence prediction:

- Countries with the same fundamentals $(s, \delta, n, \gamma, \alpha)$ converge to the *same* balanced growth path.
- If a country starts with lower k_0, it grows faster initially.

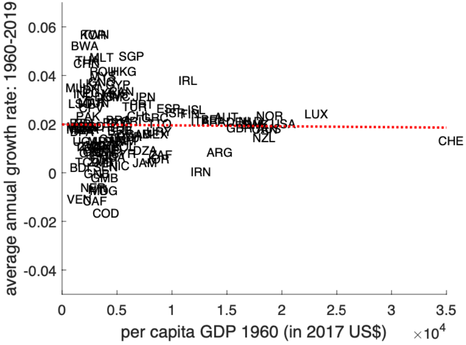

Figure: Per Capita GDP 1960 (Source: Penn World Tables 10.0)

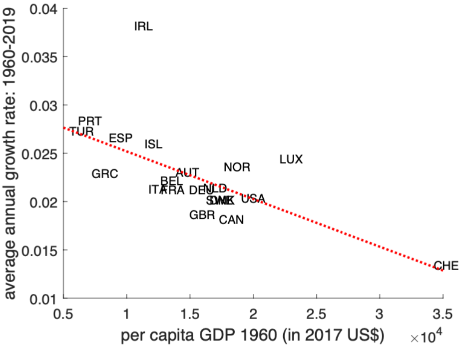

Figure 3.7: OECD countries, 1960–2019

Figure: OECD Countries, 1960-2019 (Source: Penn World Tables 10.0)

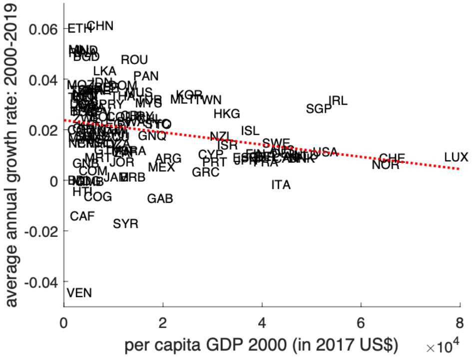

Figure: per capita GDP 2000 (in 2017 US Dollars, 1960-2019 (Source: Penn World Tables 10.0)

Speed of Convergence

Around the steady state, one can approximate:

$$\Delta k_{t+1} \approx \left[1 - \delta + s\, f'(k)\right] \Delta k_t .$$

For Cobb–Douglas $f(k) = k^{\alpha}$:

$$\Delta k_{t+1} = \left[1 - \delta(1-\alpha)\right] \Delta k_t .$$

Define $\lambda = \delta(1-\alpha)$. Then

$$k_{t+1} - \bar{k} \;\approx\; (1-\lambda)\left(k_t - \bar{k}\right).$$

λ is the *speed of convergence*. Empirically, Solow's baseline might over-predict convergence speed, so modifications (human capital, etc.) can reconcile this.

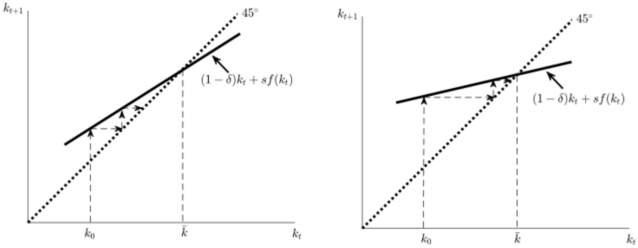

Figure 3.5: Slow and fast convergence

Cross-Country Evidence

- **Unconditional** convergence: All countries do not converge to the same path because structural parameters differ.
- **Conditional** convergence: Similar countries (e.g., same $s, \alpha, \ldots$) do converge.
- Empirical findings (e.g., OECD or US states) show faster growth where initial y_0 is lower, consistent with **conditional** convergence.
- Recent data suggests some unconditional convergence in certain periods (Kremer et al. 2020).

Using Solow Quantitatively

- **Calibration**:
  - Assume Cobb–Douglas: $F(K, AL) = K^{\alpha}(AL)^{1-\alpha}$.
  - Use micro/aggregate data to select $\alpha, \delta, n, \gamma, \ldots$.
  - Typical benchmark: $\alpha \approx 1/3,\ \delta \approx 0.05,\ n \approx 0.01,\ \gamma \approx 0.02$.
- **Speed of convergence:** $\lambda = \delta(1-\alpha) \approx 0.05 \times (2/3) \approx 0.033$, a bit high relative to cross-country data (which is often 0.02 or lower).
- **Policy experiments:** E.g., doubling s from 0.1 to 0.2 raises steady-state k and y by some

Solow Foundations for Business Cycles

- Modern RBC and New Keynesian models build on the core $k_{t+1} = s\,F(k_t, l_t) + \ldots$.
- Business cycles as deviations from a growth path:

$$A_t \;\text{(tech shocks)}, \quad \nu_t \;\text{(investment-specific)}, \quad c_t \;\text{(demand shocks)}, \ldots$$

- Studying *impulse responses* in (k, y, c) to these shocks helps us understand dynamic adjustment mechanisms.

Impulse Responses

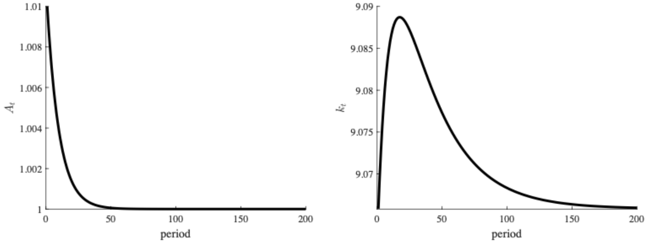

Figure 3.9: Impulse response: how A_t (left) affects k_t (right)

Example: Technology Shock

$$k_{t+1} = s\,A_t\,F(k_t, l) + (1-\delta)k_t,$$

- Suppose A_t jumps above its mean for a few periods.
- Then $\{k_t\}$ transitions to a new path but eventually returns if the shock is temporary.
- **RBC idea:** Temporary productivity shifts can cause *fluctuations* around the trend.

Impulse-Response Illustration

- If A_t follows a first-order autoregression with persistence $\rho \in (0, 1)$:

$$A_t = \bar{A}(1 + \varepsilon_t), \quad \varepsilon_{t+1} = \rho\,\varepsilon_t + \ldots$$

- Then log-linearize around steady state to get approximate solutions for how k_t and y_t respond over time.
- Typically, capital adjusts more gradually than A_t.
- $\Rightarrow$ RBC or DSGE models use such linearization to derive *impulse-response functions* (IRFs).

Summary

- The **Solow model** provides a framework for:
  - Explaining why capital-output ratio can remain stable.
  - Understanding role of s, δ, n, γ in steady-state levels and growth.
- **Key predictions**:
  - Long-run per-capita growth requires technological progress.
  - Conditional convergence among similar economies.
- Forms the **foundation** for growth and business-cycle theories.

Further Reading

- **Barro, R. and Sala-i-Martin, X.** (2004). *Economic Growth.* 2nd edition. MIT Press. (Barro and Sala-i Martin, 1992)
- **Cooley, T.F. & Prescott, E.C.** (1995). "Economic Growth and Business Cycles," in *Frontiers of Business Cycle Research*. (Cooley and Prescott, 1995)
- **Kremer, M. et al.** (2020). "Unconditional Convergence in the 21st Century?" (Kremer et al., 2020)
- **Uzawa, H.** (1961). "Neutral Inventions and the Stability of Growth Equilibrium." *Review of Economic Studies.* (Uzawa, 1961)

Appendix: (Uzawa, 1961) Theorem Sketch

Part I: If an economy's production side has constant returns to scale and admits a strictly positive balanced growth path for capital and output, it must be that technical progress is labor-augmenting.
Part II: Under standard output/investment identities ($Y_t = C_t + I_t$, $K_{t+1} - K_t = I_t - \delta K_t$), any constant growth of Y_t, C_t, I_t, K_t must have $\gamma_Y = \gamma_K$.
Hence labor-augmenting (Harrod-neutral) is the only form consistent with exact balanced growth in the long run.

References

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Thank you!

Questions or comments?

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