

Chapter 4: Dynamic Optimization

Graduate Macroeconomics Slides

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Motivation

- In Chapter 3 (Solow model), saving/investment decisions were *exogenously* given via a constant saving rate.
- (Cass, 1965) and (Koopmans, 1963) introduced microfoundations for saving, making it an *endogenous* choice.
- These optimizing foundations gave rise to the modern *neoclassical growth model*.
- Adding **dynamic optimization** and **forward-looking** behavior greatly enriches both positive and normative analysis.



Key Questions

- How do we incorporate intertemporal trade-offs in macroeconomic models?
- Finite vs. infinite horizons: Does it matter for real-world policy and predictions?
- What is the **Euler Equation** and why is it central to dynamic problems?
- How do **dynamic programming methods** simplify solving complex infinite-horizon problems?



Setup of a Finite-Horizon Problem

Generic form:

$$\max_{\{y_t, x_{t+1}\}_{t=0}^T} \sum_{t=0}^T \beta^t \hat{F}(y_t)$$

subject to

$$x_{t+1} = h(x_t, y_t), \quad x_{t+1} \in \Gamma(x_t), \quad x_0 \text{ given.}$$

- x_t : *state variable* (e.g. capital stock).
- y_t : *control variable* (e.g. consumption, saving).
- $\hat{F}(y_t)$: one-period objective (e.g. utility).
- β : discount factor, $0 < \beta < 1$.
- T : finite number of periods.



Example: 2-Period Consumption-Saving

- Household lives for $T = 2$ periods.
- Budget constraints:

$$c_0 = w_0 - a_1, \quad c_1 = w_1 + (1 + r)a_1,$$

- Utility:

$$\max_{c_0, c_1, a_1} u(c_0) + \beta u(c_1).$$

- Non-negativity constraint: $a_1 \geq 0$ and $c_t \geq 0$.
- Lagrangian approach \Rightarrow **Euler Equation:**

$$u'(c_0) = \beta(1 + r) u'(c_1).$$



Neoclassical Growth Model (Finite T)

$$\max_{\{k_{t+1}\}_{t=0}^T} \sum_{t=0}^T \beta^t u(f(k_t) + (1 - \delta)k_t - k_{t+1}),$$

subject to

$$k_{t+1} \geq 0, \quad k_0 \text{ given}, \quad t = 0, \dots, T.$$

Key FOCs

$$u'(c_t) = \beta u'(c_{t+1}) [f'(k_{t+1}) + 1 - \delta], \quad c_t = f(k_t) + (1 - \delta)k_t - k_{t+1}.$$

Terminal condition: $k_{T+1} = 0$ (often arises naturally in finite horizons).



Backward Induction

1. Start from last period T : set $k_{T+1} = 0$ in FOC.
2. Solve for k_T from Euler equation for $t = T - 1$.
3. Move to $t = T - 2$, etc.

Example (Log Utility, Cobb-Douglas Production, $\delta = 1$):

$$u(c) = \ln(c), \quad f(k) = A k^\alpha.$$

$$k_{t+1} = \frac{\alpha\beta A}{1 + \alpha\beta} k_t^\alpha \quad \implies \quad (\text{closed-form solution for each } t).$$



Why Infinite Horizon?

- Finite horizon leaves decisions sensitive to “last period” effects (e.g. $k_{T+1} = 0$).
- Often in macro, we want steady-state or long-run implications \Rightarrow infinite horizon.
- Infinite horizon can also be interpreted as *overlapping generations* or *dynastic* settings where each generation cares about descendants.



Ponzi Schemes

- With infinite horizon, one can try to borrow repeatedly “forever” without repaying principal (*Ponzi game*).
- This can make the problem ill-defined: utility unbounded above.

No-Ponzi-Game Condition (NPG)

$$\lim_{t \rightarrow \infty} \frac{a_{t+1}}{(1+r)^t} \geq 0$$

prevents rolling over debt forever.



Consumption-Saving with $T = \infty$

$$\begin{aligned} \max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t u(c_t), \\ \text{s.t.} \quad & c_t + a_{t+1} = (1+r)a_t + w_t, \quad t = 0, 1, 2, \dots \\ & \lim_{t \rightarrow \infty} \frac{a_{t+1}}{(1+r)^t} \geq 0, \quad a_0 \text{ given.} \end{aligned}$$

Euler equation (FOC):

$$u'(c_t) = \beta(1+r)u'(c_{t+1}).$$

Key outcome: Transversality Condition (TVC)

$$\lim_{t \rightarrow \infty} \beta^t u'(c_t) a_t = 0.$$

Infinite-Horizon NGM

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) + (1 - \delta)k_t - k_{t+1}),$$

subject to

$$k_{t+1} \geq 0, \quad k_0 \text{ given, and TVC.}$$

Euler Equation:

$$u'[c_t] = \beta u'[c_{t+1}] [f'(k_{t+1}) + 1 - \delta], \quad c_t = f(k_t) + (1 - \delta)k_t - k_{t+1}.$$

Steady State

$$k^* : \quad 1 = \beta [f'(k^*) + 1 - \delta].$$



Convergence Intuition

- If $k_t < k^*$, net marginal product $> 1/\beta - (1 - \delta) \Rightarrow$ strong incentive to invest $\Rightarrow k_{t+1} > k_t$.
- If $k_t > k^*$, net marginal product $< 1/\beta - (1 - \delta) \Rightarrow k_{t+1} < k_t$.
- Globally, $\{k_t\}$ converges to k^* (assuming Inada conditions and strict concavity).



Dynamic Programming: Key Idea

Rewrite the infinite-horizon problem as:

$$V(x) = \max_{x' \in \Gamma(x)} \{ F(x, x') + \beta V(x') \}.$$

- $V(x)$: *value function* = maximum discounted payoff from state x .
- x' chosen to maximize current payoff $F(x, x')$ plus discounted future value $V(x')$.
- Once $V(x)$ is found, the optimal *policy function* $g(x) = \arg \max_{x'} \{ F(x, x') + \beta V(x') \}$ follows.



Equivalence to Sequential Formulation

- Stationary environment \implies same functional problem each period, so $V(\cdot)$ doesn't explicitly depend on t .
- Under suitable assumptions (concavity, boundedness, $\beta < 1$), there is a *unique* continuous solution $V^*(x)$.
- The sequential solution $\{x_t^*\}$ must satisfy $x_{t+1}^* = g(x_t^*)$ and the Euler equation plus TVC.



Bellman Equation for the NGM

$$V(k) = \max_{k' \in [0, f(k) + (1-\delta)k]} \left\{ u(f(k) + (1-\delta)k - k') + \beta V(k') \right\}.$$

- Sufficient conditions: $u(\cdot)$ strictly increasing, concave; $f(\cdot)$ concave, satisfies Inada conditions; $\beta \in (0, 1)$.
- Unique solution $V^*(k)$ and policy $k' = g(k)$.
- *Functional Euler Equation:*

$$u'(f(k) - g(k)) = \beta u'(f(g(k)) - g(g(k))) [f'(g(k)) + 1 - \delta].$$



Properties of $V(k)$ and $g(k)$

Under standard assumptions:

1. $V(k)$ is strictly increasing and concave.
2. Optimal policy $g(k)$ is *monotonic* (strictly increasing).
3. Convergence: $k_{t+1} = g(k_t)$ converges to the unique k^* solving $1 = \beta[f'(k^*) + 1 - \delta]$.



Iterating on the Value Function

$$V_{n+1}(x) = \max_{x' \in \Gamma(x)} \left\{ F(x, x') + \beta V_n(x') \right\}, \quad n = 0, 1, 2, \dots$$

- Start with any guess $V_0(x)$.
- Repeatedly apply the *Bellman operator* until convergence:

$$\|V_{n+1} - V^*\| \leq \beta \|V_n - V^*\|.$$

- Convergence is at geometric rate β .



Guess and Verify

- Sometimes feasible with specific forms (e.g. CRRA utility, Cobb-Douglas production).
- Example: $u(c) = \ln(c)$, $f(k) = Ak^\alpha$, $\delta = 1$.

Guess: $V(k) = a + b \ln k$.

- Plug into Bellman equation; solve for a, b (undetermined coefficients).
- Recover policy $g(k)$ from the FOC/Euler equation.



Linearization around Steady State

- Expand policy $g(k)$ via Taylor series around k^* :

$$g(k) \approx g(k^*) + g'(k^*)(k - k^*) + \dots$$

- Solve for slope $g'(k^*)$ using the *functional Euler equation* derivative at k^* .
- If $|g'(k^*)| < 1$, we get local stability; if > 1 , steady state is unstable.



Balanced Growth in the NGM

- Allow A_t to grow at rate γ , population L_t grow at n .
- Utility often becomes:

$$\sum_{t=0}^{\infty} \beta^t L_t u(c_t).$$

- Transform to per-effective-labor terms: $\tilde{k}_t = \frac{k_t}{A_t L_t}$.
- *CRRA utility* ensures balanced growth path:

$$\beta' = \beta (1 + n) (1 + \gamma)^{1-\sigma}.$$

- Steady-state condition:

$$1 = \beta' [f'(\tilde{k}^*) + 1 - \delta].$$



Phase Diagrams

- Plot $\dot{k}_t = 0$ and $\dot{c}_t = 0$ in (k, c) space (or discrete analog).
- Intersection \rightarrow steady state (k^*, c^*) .
- *Saddle path*: system typically has one stable branch (monotonically approaching equilibrium).



Policy Implications

- Endogenous saving leads to different responses to shocks than exogenous saving.
- Forward-looking behavior can amplify or dampen certain policies.
- *Real Business Cycle (RBC) framework* extends the NGM with stochastic shocks.



Empirical Evidence

- **Consumption smoothing:** Data supports Euler equations with some deviations (liquidity constraints, habits, etc.).
- **Speed of convergence:** Empirical estimates often find about 2%–3% convergence per year, though micro-level variations exist.
- **CRRA preferences:** Reasonably matches macro consumption patterns, but σ (risk aversion vs. EIS) is debated.



Limitations & Extensions

- Borrowing constraints and credit market imperfections \Rightarrow kinked Euler equations.
- Uncertainty (idiosyncratic or aggregate) \Rightarrow precautionary saving, incomplete markets.
- Overlapping generations: finite lifetime, but each generation's saving decisions shape aggregate capital.
- Behavioral: deviations from rational expectations, time-inconsistent preferences.



Summary

- **Finite-horizon models:** Last-period effects; solved by backward induction.
- **Infinite-horizon models:** Stationarity; Euler equation + Transversality condition essential.
- **Dynamic programming (Bellman):** Powerful for both theoretical insights and numerical solutions.
- **Neoclassical Growth Model:** Endogenous saving, unique steady state, global convergence under standard assumptions.



References

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Thank you!

Questions or comments?

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