Chapter 7: Uncertainty

Graduate Macroeconomics Slides

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Outline

Motivation

- Many aspects of economic life are unpredictable:
  - Technological changes
  - Policy shifts
  - Individual-level shocks (income, health, etc.)
- Goal: Introduce the main techniques for incorporating uncertainty into macroeconomic analysis.
- This chapter provides the foundational tools to handle stochastic processes, decision-making under uncertainty, and equilibrium under uncertainty.

Chapter Overview

1. **Stochastic Processes:** Key concepts, stationarity, Markov chains, AR(1) processes.
2. **Choice under Uncertainty:** Expected utility, risk aversion, portfolio choice.
3. **Stochastic Growth Model:** Planner's problem and competitive equilibrium.
4. **Complete vs. Incomplete Markets:** Risk-sharing implications.
5. **Critical Analysis:** Limitations and extensions.

Why Uncertainty Matters

- Economic agents face fundamental uncertainty:
  - At micro level: income, employment, health
  - At macro level: shocks to technology, policy, preferences
- We model these using stochastic processes and analyze behavior using:
  - Conditional expectations
  - Probability distributions
  - Decision theory under risk
- Goal: Integrate uncertainty into equilibrium frameworks like growth models or DSGE.

Stochastic Processes: Key Concepts

Let $\{X_t\}_{t \in \mathbb{Z}_+}$ be a stochastic process.

Definitions

- **Unconditional expectation:** $\mathbb{E}[X_t]$
- **Conditional expectation:** $\mathbb{E}_t[X_{t+1}] = \mathbb{E}[X_{t+1} \mid \mathcal{F}_t]$
- **Law of iterated expectations:** $\mathbb{E}_t[\mathbb{E}_s[X_\tau]] = \mathbb{E}_t[X_\tau]$ for $t < s < \tau$

Stationarity and Ergodicity

- **Covariance stationarity:** $\mathbb{E}[X_t] = \mu$, $\text{Cov}(X_t, X_{t+j})$ only depends on j
- **Ergodicity:** Time averages converge to expectations:

$$\frac{1}{T}\sum_{t=1}^{T} X_t \xrightarrow{a.s.} \mathbb{E}[X_t]$$

Markov Processes and Markov Property

A process $\{X_t\}$ is (first-order) **Markov** if:

$$\Pr(X_{t+1} = x \mid X_t, X_{t-1}, \dots) = \Pr(X_{t+1} = x \mid X_t)$$

- The entire predictive distribution of future variables depends only on the current state.
- In macro: TFP, employment status, or policy regimes are often modeled as Markov processes.

Implication: It allows us to formulate recursive problems in dynamic programming.

Markov Decision Processes (MDPs)

State X_t, action a_t, transition function $\Pr(X_{t+1} \mid X_t, a_t)$, and reward $r(X_t, a_t)$ define optimal behavior under uncertainty.

Markov Chains and Transition Matrices

Suppose $X_t \in \mathcal{X} = \{x_1, \ldots, x_N\}$ (finite state space).

- We define the **transition matrix** $P \in \mathbb{R}^{N \times N}$ where $P_{ij} = \Pr(X_{t+1} = x_j \mid X_t = x_i)$
- Each row of P sums to 1.
- Initial distribution: $\pi_0 \in \Delta^N$ (simplex)
- Law of motion: $\pi_{t+1} = \pi_t P$, so:

$$\pi_{t+k} = \pi_t P^k$$

Stationary Distribution

$\bar{\pi}$ is stationary if $\bar{\pi} = \bar{\pi} P$.
If P is irreducible and aperiodic, then $\lim_{t \to \infty} \pi_t = \bar{\pi}$ for all π_0.

Unemployment Model: Markov Chain Example

Agents can be:

- Employed (e) or Unemployed (u)

Transition matrix:

$$P = \begin{bmatrix} 1-s & s \\ f & 1-f \end{bmatrix}$$

Steady state: Let $\bar{u}$ be the unemployment rate.
Solving $\bar{\pi} P = \bar{\pi}$ with $\bar{\pi} = [1-\bar{u}, \bar{u}]$ gives:

$$\bar{u} = \frac{s}{s+f}$$

Interpretation: Fraction of people flowing in equals flowing out:

$$\underbrace{(1-\bar{u})s}_{\text{Employed}\to\text{Unemployed}} = \underbrace{\bar{u}f}_{\text{Unemployed}\to\text{Employed}}$$

Autoregressive (AR(1)) Process

Let $\{x_t\}$ follow:

$$x_t = \rho x_{t-1} + b\varepsilon_t + (1-\rho)\mu, \quad \varepsilon_t \sim \text{i.i.d. } \mathcal{N}(0,1)$$

- **Stationary** if $|\rho| < 1$
- **Mean:** $\mathbb{E}[x_t] = \mu$
- **Variance:** $\text{Var}(x_t) = \frac{b^2}{1-\rho^2}$
- **Autocovariance:** $\text{Cov}(x_t, x_{t+j}) = \frac{b^2\rho^j}{1-\rho^2}$

Uses: Business cycles, productivity, inflation.
Remark: AR(1) can be derived from a simple linearization of DSGE models.

Linear Stochastic Difference Equations

Let $x_t \in \mathbb{R}^n$, $\varepsilon_t \in \mathbb{R}^m$. Suppose:

$$x_t = Ax_{t-1} + B\varepsilon_t + C$$

Assume:

$$\mathbb{E}[\varepsilon_t] = 0, \quad \mathbb{E}[\varepsilon_t \varepsilon_t'] = I, \quad \text{Cov}(\varepsilon_t, \varepsilon_{t+s}) = 0$$

- **Unconditional mean:** $\mu = \mathbb{E}[x_t] = (I - A)^{-1} C$
- **Unconditional covariance:**

$$\Gamma(0) = A\Gamma(0)A' + BB'$$

- This is a Lyapunov equation.

Impulse Response Function

If ε_t is a one-time shock at t:

$$x_{t+h} - \mu = A^h B\varepsilon_t$$

Then $\mathcal{F}(h) = A^h B$ traces the impact path.

Event Trees for Uncertainty

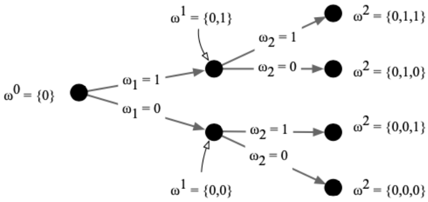


Figure: Figure 7.1: Example event tree

- Each path ω^t defines a history up to period t.
- Tree structures help define filtrations $\mathcal{F}_t$.
- Useful for dynamic programming, Bellman recursion, or Epstein-Zin utility.

Modeling Uncertainty

- We represent uncertainty via *stochastic events* $c_t \in \Omega_t$, with $\pi_t(c_t)$ as the probability of event c_t.
- The history up to date t: $c^t = (c_0, c_1, \ldots, c_t)$.
- Allocations and decisions (consumption, assets, etc.) can depend on the realized history c^t.

Expected Utility and Risk Aversion

- **Expected Utility:** $E[\sum_{t=0}^{\infty} \beta^t u(c_t)]$.
- **Risk aversion** arises if $u(\cdot)$ is concave.
- **Absolute vs. Relative Risk Aversion (ARA, RRA):**

$$ARA = -\frac{u''(c)}{u'(c)}, \quad RRA = -\frac{c\, u''(c)}{u'(c)}.$$

- **CRRA utility:** $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ has constant RRA $= \sigma$.
- Smoother utility (larger σ) $\implies$ more risk averse.

Portfolio Choice Example

- Wealth W: invest A in a risky asset with random return Z, and $(W - A)$ in a safe asset with return R_f.
- Consumption next period: $C_1(\omega) = R_f(W - A) + Z(\omega)\, A$.
- **Optimality condition** (CRRA case):

$$E_0 \left[u'(C_1(\omega)) \left(Z(\omega) - R_f \right) \right] = 0.$$

- CRRA $\implies$ fraction A/W is constant w.r.t. W.

Two-Period Stochastic Growth Model

Setup:

- TFP in period 1 is random, $A_1(c_1)$ with probability $\pi_1(c_1)$.
- Household utility:
$$U = u(C_0) + \beta \sum_{c_1} \pi_1(c_1)\, u\big(C_1(c_1)\big).$$
- Resource constraints:
$$K_1 + C_0 = K_0^{\alpha} + (1-\delta)K_0, \quad C_1(c_1) = A_1(c_1)\, K_1^{\alpha} + (1-\delta)K_1.$$

Two-Period Planner's Problem

$$\max_{\{C_0, K_1, C_1(c_1)\}} \quad u(C_0) \; + \; \beta \sum_{c_1} \pi_1(c_1)\, u\big(C_1(c_1)\big)$$

subject to the resource constraints.

Optimality (Euler) condition:

$$u'(C_0) = \beta \sum_{c_1} \pi_1(c_1)\, u'\big(C_1(c_1)\big)\, \big[\alpha\, A_1(c_1)\, K_1^{\alpha-1} + 1 - \delta\big].$$

Interpretation: marginal utility of saving 1 unit now equals the expected marginal utility gain next period.

Infinite-Horizon Stochastic Growth Model

- Continuum of periods $t = 0, 1, 2, \ldots$.
- TFP $A_t(c_t)$; assume it follows a Markov process.
- Production function $Y_t(c_t) = A_t(c_t)\, F(K_t(c_{t-1}),\, 1)$.
- Resource constraint:

$$K_{t+1}(c_t) + C_t(c_t) = F\big(K_t(c_{t-1}),\, 1\big)\, A_t(c_t) + (1-\delta)\, K_t(c_{t-1}).$$

- Expected utility:

$$\sum_{t=0}^{\infty} \sum_{c_t} \beta^t\, \pi_t(c_t)\, u\big(C_t(c_t)\big).$$

Planner's First-Order Conditions

Stochastic Euler Equation

$$u'(C_t(c_t)) = \beta \sum_{c_{t+1}|c_t} \pi_{t+1}(c_{t+1} \mid c_t)\, u'\left(C_{t+1}(c_{t+1})\right) \left[F_K(A_{t+1}(c_{t+1}),\ K_{t+1}(c_t)) + 1 - \delta\right].$$

- Marginal utility of consumption today vs. *expected* marginal utility of future consumption times the marginal product of capital.
- Risk matters because C_{t+1} and A_{t+1} are uncertain.
- If we define an expectation operator: $E_t[\cdot] \equiv \sum \pi_{t+1}(c_{t+1} \mid c_t) \ldots$, the equation becomes

$$u'(C_t) = \beta\, E_t\left[u'(C_{t+1})\,(F_K(A_{t+1}, K_{t+1}) + 1 - \delta)\right].$$

Recursive Formulation

- Let $V(A, K)$ be the value function; TFP A follows a Markov chain.

$$V(A, K) = \max_{C, K'} \{ u(C) + \beta \sum_{A'} \pi(A'|A) \, V(A', K') \}$$

subject to

$$K' + C = f(A, K) \equiv A \, F(K, 1) + (1 - \delta)K.$$

- The solution yields the same allocation as the sequential planner's problem.
- (Brock and Mirman, 1972) is a seminal reference on stochastic growth in a planning framework.

Linear Approximation Approach

- Assume A_t follows AR(1): $A_{t+1} = \rho A_t + (1-\rho)\bar{A} + \varepsilon_{t+1}$.
- Log-linearize around the deterministic steady state $(\bar{K}, \bar{A})$.
- Yield approximate policy functions:

$$\hat{K}_{t+1} = g_K\,\hat{K}_t + g_A\,\hat{A}_t, \quad \hat{A}_{t+1} = \rho\,\hat{A}_t + \varepsilon_{t+1}.$$

- **Certainty Equivalence** in the linearized model: only *expected* shocks matter, not higher moments like variances.

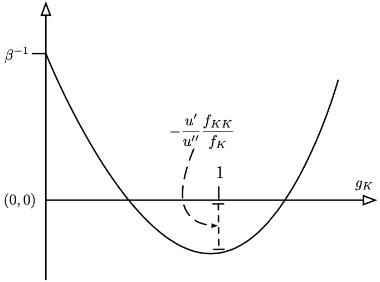


Figure 7.2: Quadratic equation to determine g_K in the linearized stochastic growth model.

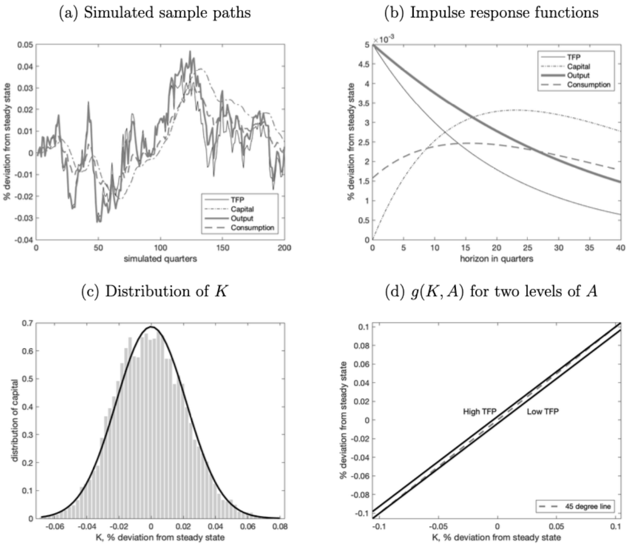
Illustration of Impulse Responses

- A one-time shock ε_t raises A_t.
- Immediate effect: output rises proportionally.
- Over time, capital accumulates more, so Y_{t+h} can exceed the direct effect of the shock as it feeds into higher K.
- Consumption also responds but is smoothed across periods due to diminishing marginal utility.

Complete vs. Incomplete Markets

- **Complete markets:** Agents can insure against all states of the world (there is a contract or asset for every contingency).
- **Incomplete markets:** Some state-contingent claims are missing. Agents cannot fully insure each other.
- With complete markets, each agent's consumption depends only on *aggregate* shocks (full idiosyncratic risk-sharing).
- With incomplete markets, individual shocks matter for each agent's consumption path.

Competitive Market Trade Under Uncertainty



Complete vs. Incomplete Markets

- **Complete Markets:**
  - Agents trade contracts for *every possible state* ω^t.
  - Arrow-Debreu securities: Pay 1 unit in specific (t, ω^t).
  - Full insurance possible (idiosyncratic risks eliminated).
- **Incomplete Markets:**
  - Limited contracts (some risks uninsurable).
  - Focus: Complete markets for simplicity (Section 7.6 covers incomplete).

Arrow-Debreu Competitive Equilibrium

Household Problem: Maximize expected utility:

$$\sum_{t=0}^{\infty} \sum_{\omega^t} \beta^t \pi(\omega^t) u(c_{i,t}(\omega^t))$$

subject to budget:

$$\sum_{t=0}^{\infty} \sum_{\omega^t} p_t(\omega^t) c_{i,t}(\omega^t) = \sum_{t=0}^{\infty} \sum_{\omega^t} p_t(\omega^t) y_{i,t}(\omega^t)$$

Equilibrium Definition:

- Prices $\{p_t(\omega^t)\}$ and allocations $\{c^*_{i,t}(\omega^t)\}$.
- Markets clear: $\sum_i c^*_{i,t} = \sum_i y_{i,t} \; \forall t, \omega^t$.

Risk Sharing & First-Order Conditions

Key FOC:

$$\beta^t \pi(\omega^t) u'(c_{i,t}(\omega^t)) = \lambda_i p_t(\omega^t)$$

- **Actuarially Fair Prices:** $p_t(\omega^t) \propto \pi(\omega^t)$.
- Full insurance: $c_{i,t}(\omega^t)$ constant across ω^t (if aggregate resources permit).

Risk Sharing Ratio:

$$\frac{u'(c_{i,t})}{u'(c_{j,t})} = \frac{\lambda_i}{\lambda_j} \quad \forall i, j$$

Consumption depends on aggregate shocks only.

Aggregate vs. Idiosyncratic Risks

- **Aggregate Risks:** Affect total resources (e.g., GDP shocks).
- **Idiosyncratic Risks:** Affect individuals (e.g., unemployment).
- Law of Large Numbers (LLN):
  - With continuum $\mathcal{I} = [0, 1]$, idiosyncratic risks cancel out.
  - Aggregate resources deterministic under LLN.
- Example: Unemployment shocks with random π_e (aggregate risk).

Sequential Trading with Arrow Securities

Budget Constraint:

$$c_{i,t}(\omega^t) + \sum_{\omega_{t+1}} q_t(\omega_{t+1}|\omega^t) a_{i,t+1}(\omega^{t+1}) \leq y_{i,t}(\omega^t) + a_{i,t}(\omega^t)$$

- Arrow securities: Price $q_t(\omega_{t+1}|\omega^t)$ for next-period payoffs.
- No Ponzi condition: $a_{i,t} \geq -\sum_{\tau=t}^{\infty} \tilde{q}_\tau^t y_{i,\tau}$.

Sequential Equilibrium:

- Allocation matches Arrow-Debreu equilibrium.
- Prices linked via $\tilde{q}_\tau^t = \prod_{s=t}^{\tau-1} q_s(\omega_{s+1}|\omega^s)$.

Spanning & Market Completeness

Payoff Matrix & Completeness:

- S states, N assets. Payoff matrix $D \in \mathbb{R}^{S \times N}$.
- Complete markets $\iff$ $\text{rank}(D) = S$.
- Arrow securities span payoff space: $D\theta =$ any $z \in \mathbb{R}^S$.

Key Result:

Markets are complete if assets can replicate Arrow securities.

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- Example: Unemployment shocks with random π_e (aggregate risk).

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Budget Constraint:

$$c_{i,t}(\omega^t) + \sum_{\omega_{t+1}} q_t(\omega_{t+1}|\omega^t) a_{i,t+1}(\omega^{t+1}) \leq y_{i,t}(\omega^t) + a_{i,t}(\omega^t)$$

- Arrow securities: Price $q_t(\omega_{t+1}|\omega^t)$ for next-period payoffs.
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Spanning & Market Completeness

Payoff Matrix & Completeness:

- S states, N assets. Payoff matrix $D \in \mathbb{R}^{S \times N}$.
- Complete markets $\iff$ rank$(D) = S$.
- Arrow securities span payoff space: $D\theta =$ any $z \in \mathbb{R}^S$.

Key Result:

Markets are complete if assets can replicate Arrow securities.

Arrow-Debreu Equilibrium (Complete Markets)

- Trade occurs at $t = 0$ with contracts specifying delivery of goods contingent on each possible future state c^t.
- **Arrow security prices:** $p_t(c^t)$ is the cost in date-0 goods of one unit of date-t goods if c^t occurs.
- **Household budget constraint:**

$$\sum_{t,c^t} p_t(c^t)\, c_{i,t}(c^t) \;=\; \sum_{t,c^t} p_t(c^t)\, y_{i,t}(c^t).$$

- **Equilibrium:** Households choose $c_{i,t}(c^t)$ to maximize utility subject to the budget constraint; markets clear ($\sum_i c_{i,t}(c^t) = \sum_i y_{i,t}(c^t)$).

Implications of Complete Markets

- **Full insurance of idiosyncratic risk:**

$$c_{i,t}(c^t) = c_{i,t}(c'^t) \quad \text{if only individual states differ.}$$

- Only *aggregate* states matter for consumption.
- Pricing of Arrow securities is based on marginal rates of substitution across states.
- In the stochastic growth model with *representative agent* and complete markets, the equilibrium allocation coincides with the planner's solution (*First Welfare Theorem*).

Sequential Trading Implementation

- Instead of trading all state-contingent claims at $t = 0$, one can replicate the complete set of Arrow securities by trading one-period assets *every* period.
- **Key condition:** The available assets must *span* all payoff vectors across states.
- The same equilibrium allocation emerges whether we use date-0 Arrow-Debreu securities or repeated one-period Arrow securities (*equivalence*).

Household and Firm Problems

Household:

$$\max \sum_{t=0}^{\infty} \beta^t \pi_t(c_t)\, u(c_t(c_t)),$$

subject to

$$c_t(c_t) + k_{t+1}(c_t) \;=\; (r_t(c_t) + 1 - \delta)k_t(c_{t-1}) + w_t(c_t).$$

Firm:

$$\max_{k_t,\, l_t} \;\; A_t(c_t)F(k_t,\, l_t) - r_t(c_t)k_t - w_t(c_t)\, l_t,$$

which yields

$$r_t(c_t) = A_t F_K(k_t, 1), \quad w_t(c_t) = A_t F_L(k_t, 1).$$

Result: Equivalence with the Planner's Allocation

- By substituting r_t and w_t from the firm's problem into the household budget constraint, we recover the same resource constraint as the planner's.
- **Euler equation**:

$$u'(c_t) = \beta\, E_t\left[u'(c_{t+1})\,(r_{t+1} + 1 - \delta)\right].$$

- Since $r_{t+1} = A_{t+1} F_K(k_{t+1}, 1)$, this matches the planner's first-order condition.
- Hence, the competitive equilibrium allocation = solution to the social planner's problem.

Motivation for Incomplete Markets

- Many risks in real economies are *not* fully insurable (e.g., private information, moral hazard, limited contract enforcement).
- Households can only self-insure via savings in a single asset (e.g., a risk-free bond).
- Implications for consumption and wealth distributions, business cycles, policy, etc.

A Simple Incomplete-Market Setup

- Income y_t follows some Markov process with states $\{y_1, \ldots, y_m\}$.
- One risk-free asset with gross return $R = 1 + r$.
- Budget constraint:

$$a_{t+1} + c_t = a_t R + y_t, \quad a_{t+1} \geq a_{\min}.$$

- Household problem:

$$V(a, y) = \max_{a'} \Big\{ u(R\,a + y - a') + \beta\, E\big[V(a', y') \mid y\big] \Big\}.$$

- No perfect insurance $\implies$ consumption still depends on idiosyncratic y_t.

(a) Decision rule for net saving $g(a, y) - a$

(b) Simulated stationary distribution of assets

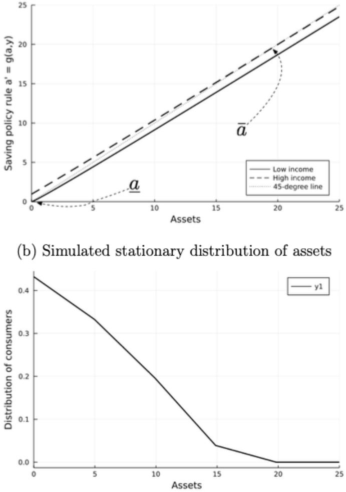

Figure 7.4: Decisions rules and stationary distribution for the incomplete-market consumption-savings problem.

Qualitative Features

- Households smooth consumption *partially* by adjusting assets a_t.
- If y_t is temporarily low, consumption will drop but not necessarily by the full drop in income.
- Stationary asset distribution emerges in the limit for large t if $\beta R < 1$ or under mild conditions.
- This distribution can exhibit interesting inequalities in consumption and wealth.

Critical Analysis

- **Representative-agent vs. heterogeneous agents:**
  - RA models typically assume complete markets.
  - Heterogeneous-agent models may have incomplete markets and yield different policy implications.
- **Certainty equivalence in linearized models:** Neglects higher-order terms and precautionary savings motives.
- **Empirical evidence:** Real economies face incomplete markets—can the RBC approach (stochastic growth with complete markets) match data on wealth and consumption distribution?

Further Research Directions

- Extending the stochastic growth model to:
  - *Heterogeneous* agents with incomplete markets
  - Additional frictions (e.g., adjustment costs, liquidity constraints)
  - Endogenous labor supply and labor market uncertainty
- Non-linear solution methods vs. linear approximations
- Empirical calibration and estimation of stochastic growth models

References I

Brock, W. A. and Mirman, L. J. (1972). Optimal economic growth and uncertainty: The discounted case. *Journal of Economic Theory*, 4:479–513.

Thank you!

Questions or comments?

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