

Chapter 9: Continuous Time Analytical Techniques

Graduate Macroeconomics Slides

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January 23, 2026



Outline

- Introduction and Overview
- Continuous-Time Analytical Techniques
 - Basic Continuous-Time Notation and Growth Rates
 - Optimization in Continuous-Time Models: Maximum Principle
 - Continuous-Time Growth Models
 - Poisson Process and Jumps
- Conclusion



Chapter 9: Tools – Overview

- This chapter presents mathematical and computational tools that complement earlier chapters.
- Emphasizes **continuous-time** analytical techniques and **computational methods**:
 - Continuous-time notation and growth rates
 - Maximum Principle (optimal control)
 - Continuous-time growth models
 - Introduction to Poisson processes (jump processes)
 - Numerical methods: root-finding, optimization, discretization
 - Dynamic programming and linear rational expectations models
- Goal is to equip students with rigorous tools for advanced macroeconomic research and modeling.



Continuous-Time Notation

Notation Switch:

- Discrete-time variable: X_t for $t = 0, 1, 2, \dots$
- Continuous-time variable: $X(t)$ for $t \in \mathbb{R}_+$

Time Derivative:

$$\frac{dX(t)}{dt} = \dot{X}(t)$$

- Also called *dot notation* or *Newton's notation*.
- In economics, it is standard for expressing time derivatives.



Constant Growth in Continuous Time

- In discrete time, constant growth rate γ implies:

$$X_{t+1} = (1 + \gamma)X_t \implies X_t = (1 + \gamma)^t X_0.$$

- In continuous time, the growth rate equation:

$$\frac{\dot{X}(t)}{X(t)} = \gamma \implies X(t) = e^{\gamma t} X(0).$$



Logarithmic Differentiation

- A key trick:

$$\frac{\dot{X}(t)}{X(t)} = \frac{d}{dt} [\log X(t)].$$

- Important implication: if $\log X(t)$ is a *straight line* in t , $X(t)$ grows at a *constant* rate.
- Often used in growth accounting and time-series plots (log-scale).



Maximum Principle (Optimal Control)

Finite-Horizon Example:

$$\max_{\{c_t, a_{t+1}\}} \sum_{t=0}^T \beta^t u(c_t)$$

subject to

$$a_{t+1} = w + (1 + r)a_t - c_t, \quad a_{T+1} \geq 0.$$

- **Control variable:** c_t (decided in period t).
- **State variable:** a_t (carried over from $t - 1$).



Hamiltonian Formulation (Discrete-Time Intuition)

- Lagrangian approach:

$$\mathcal{L} = \sum_{t=0}^T \left[\beta^t u(c_t) + \mu_t ((1+r)a_t + w - c_t - a_{t+1}) \right] + \lambda a_{T+1}.$$

- *Costate variable* or multiplier: μ_t .
- Hamiltonian for period t :

$$H_t = \beta^t u(c_t) + \mu_t (ra_t + w - c_t).$$

- Control FOC: $\partial H_t / \partial c_t = 0 \Rightarrow \beta^t u'(c_t) = \mu_t$.
- State FOC: $\partial H_{t+1} / \partial a_{t+1} - \mu_t = 0 \Rightarrow \mu_t = (1+r)\mu_{t+1}$.



Continuous-Time Discounting and the Discount Rate ρ

- **Discrete-Time Discounting:** Discount factor $\beta = \frac{1}{1+\rho}$ for one period.

For t periods: $\beta^t = \left(\frac{1}{1+\rho}\right)^t$.

- **Splitting Periods into Subintervals:** Subperiod length Δ . Discount rate per subperiod: $\rho\Delta$.

Discounting t periods ($= t/\Delta$ subperiods):

$$f(\Delta, t) = \left(\frac{1}{1 + \rho\Delta}\right)^{\frac{t}{\Delta}}$$



- **Continuous-Time Limit ($\Delta \rightarrow 0$):**

$$\lim_{\Delta \rightarrow 0} f(\Delta, t) = \lim_{\Delta \rightarrow 0} \left[(1 + \rho\Delta)^{1/(\rho\Delta)} \right]^{-\rho t} = e^{-\rho t}$$

Key Result

Continuous-time discounting formula:

$$e^{-\rho t} \quad (\text{Analogous to discrete } \beta^t \text{ as } \Delta \rightarrow 0)$$



Euler Equation and TVC (Discrete Time)

- From $\beta^t u'(c_t) = \mu_t$ and $\mu_t = (1+r)\mu_{t+1}$,

$$u'(c_t) = \beta(1+r) u'(c_{t+1}).$$

- *Transversality Condition (TVC):*

$$\lim_{T \rightarrow \infty} \beta^T u'(c_T) a_{T+1} = 0 \iff \lim_{T \rightarrow \infty} \frac{a_{T+1}}{(1+r)^T} = 0.$$

- Together, they characterize optimal $\{c_t, a_t\}$ paths.



Continuous-Time Problem

Setup:

$$\max_{c(t), a(t) \geq 0} \int_0^{\infty} e^{-\rho t} u(c(t)) dt$$

subject to

$$\dot{a}(t) = w + r a(t) - c(t), \quad \lim_{T \rightarrow \infty} e^{-rT} a(T) \geq 0.$$



Hamiltonian (Continuous-Time)

- Define:

$$H(t) = e^{-\rho t} u(c(t)) + \mu(t) [r a(t) + w - c(t)].$$

- **Control FOC:** $\partial H / \partial c = 0$:

$$e^{-\rho t} u'(c(t)) = \mu(t).$$

- **Costate (State) FOC:** $\dot{\mu}(t) = -\frac{\partial H}{\partial a(t)}$:

$$\dot{\mu}(t) = -\mu(t)r \quad \Rightarrow \quad \frac{\dot{\mu}(t)}{\mu(t)} = -r.$$



Euler Equation (Continuous-Time)

- From $\dot{\mu}(t)/\mu(t) = -r$ and $e^{-\rho t} u'(c(t)) = \mu(t)$,

$$\frac{d}{dt} [\log u'(c(t))] = \frac{\dot{u}'(c(t))}{u'(c(t))} = -\rho + \frac{\dot{c}(t)}{c'(t)} \times \dots$$

- **Result:**

$$\dot{c}(t) = \frac{1}{\text{CRRRA}} [r - \rho]. \quad (\text{For CRRRA utility, e.g., } u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}.)$$

- Analogue to the discrete-time Euler equation:

$$\dot{c}(t) = \frac{1}{\sigma} [r - (\rho + \delta)], \quad \dots$$

- *Transversality Condition:*

$$\lim_{T \rightarrow \infty} e^{-\rho T} u'(c(T)) a(T) = 0.$$



Solow Model in Continuous Time

- Production function: $Y(t) = F(K(t), A(t)L(t))$ with CRS.
- Labor grows at rate n : $\dot{L}(t)/L(t) = n$.
- Technology grows at rate γ : $\dot{A}(t)/A(t) = \gamma$.
- Capital accumulation:

$$\dot{K}(t) = sF(K(t), A(t)L(t)) - \delta K(t).$$



Solow Model in Continuous Time

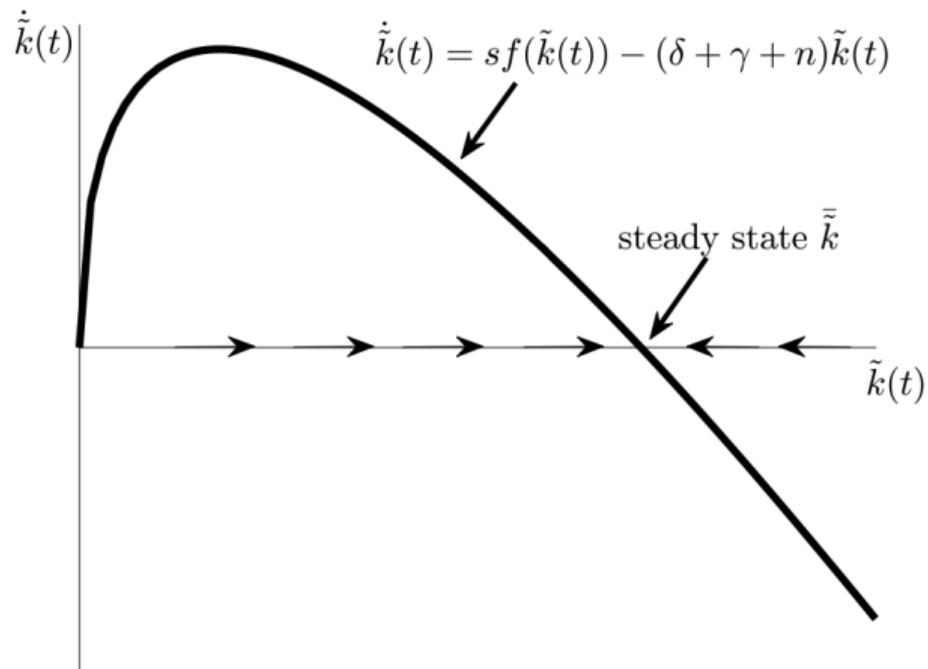


Figure 9.1: Solow model in continuous time



Solow Model: Key Equation

- Define $k(t) = K(t)/(A(t)L(t))$.
- Growth rate approach yields:

$$\dot{k}(t) = sf(k(t)) - (\delta + n + \gamma)k(t).$$

- Steady state:

$$\dot{k}(t) = 0 \Rightarrow k^* \text{ solves } sf(k^*) = (\delta + n + \gamma)k^*.$$

- Long-run balanced growth: $k(t) \rightarrow k^*$, $\frac{Y(t)}{L(t)}$ grows at rate γ .



Neoclassical Growth Model: Ramsey Setup (Continuous-Time)

- Planner's problem:

$$\max_{c(t), k(t) \geq 0} \int_0^{\infty} e^{-\rho t} u(c(t)) dt$$

- Resource constraint:

$$\dot{k}(t) = f(k(t)) - \delta k(t) - c(t).$$

- Euler equation:

$$\dot{c}(t) = \frac{1}{\sigma} [f'(k(t)) - (\delta + \rho)], \quad (\text{for CRRA utility})$$



Phase Diagram

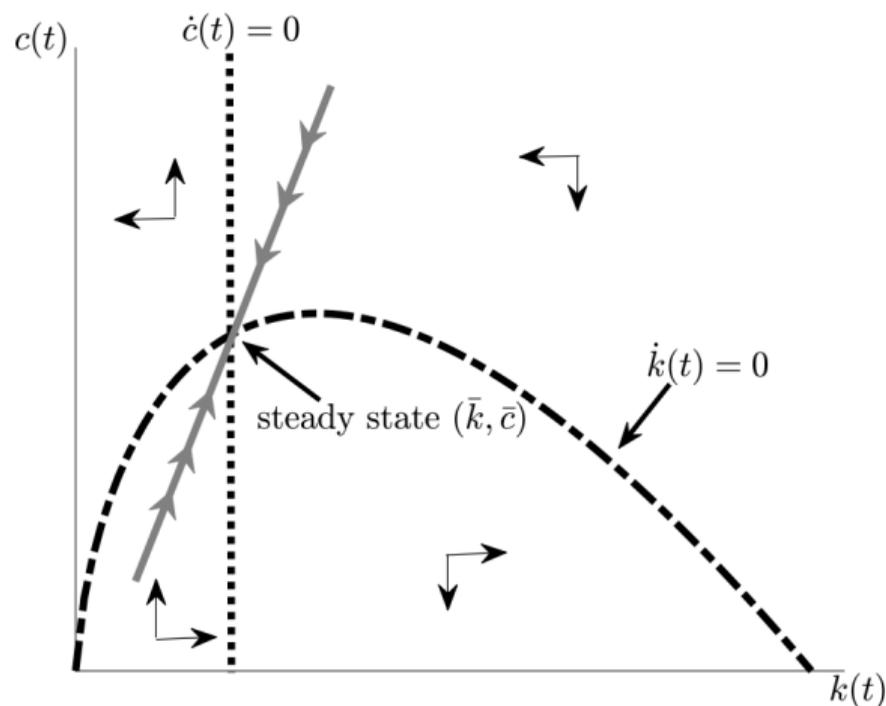


Figure 9.2: Phase diagram for Ramsey model in continuous time



Market Equilibrium Equivalence

- Competitive equilibrium solution matches planner solution under typical assumptions:

$$r(t) = f'(k(t)), \quad w(t) = f(k(t)) - k(t)f'(k(t)).$$

- Households solve continuous-time consumption-saving problem, leading to same Euler equation:

$$\dot{c}(t) = \frac{1}{\sigma} [r(t) - (\delta + \rho)].$$

- *First Welfare Theorem* ensures equivalence with the social planner outcome.



Introducing Uncertainty via Poisson (Jump) Processes

- Continuous-time uncertainty can be *diffusion* (small frequent shocks) or *jump processes* (large, infrequent shocks).
- **Poisson process:** Probability of k events in time interval $[t, t + T]$ follows

$$p(k) = \frac{e^{-\lambda T} (\lambda T)^k}{k!}.$$

- *Memoryless* process: distribution of future events independent of past realizations.
- Applications in job arrivals, default events, large discrete shocks in economics, etc.



Uncertainty in Continuous Time: Poisson Process

- **Two Types of Shocks:**
 - **Diffusion processes:** Continuous, small shocks (e.g., Brownian motion).
 - **Jump processes:** Discrete, infrequent shocks (e.g., Poisson process).
- **Bernoulli Trials to Poisson Process:**
 - Split time into n subperiods with success probability λ/n .
 - Expected successes: λ (fixed) as $n \rightarrow \infty$.
 - Binomial distribution \rightarrow Poisson distribution:

$$p(k) = e^{-\lambda} \frac{\lambda^k}{k!}$$



Deriving the Poisson Distribution

Key Steps:

- **Limit of Binomial Distribution:**

$$\lim_{n \rightarrow \infty} b(0, n) = e^{-\lambda} \quad (\text{Probability of 0 successes})$$

- **Recursive Ratio:**

$$\lim_{n \rightarrow \infty} \frac{b(k, n)}{b(k-1, n)} = \frac{\lambda}{k}$$

- **Final Poisson Formula:**

$$p(k) = e^{-\lambda T} \frac{(\lambda T)^k}{k!} \quad (\text{Events in time } T)$$

Memoryless Property

Outcomes in disjoint time intervals are independent.

Time-varying $\lambda(t)$: Probability of 0 events in $[t, t + T]$ is $e^{-\int_t^{t+T} \lambda(s) ds}$.

Example: Population Dynamics with Poisson Processes

Birth-Death Model:

- **Deaths:** Follow Poisson process with rate $\delta > 0$.
 - Survival probability up to t : $e^{-\delta t}$.
 - Expected remaining lifespan: $\frac{1}{\delta}$.
- **Births:** γdt new animals in time dt .

Population Equation:

$$\dot{n}(t) = \gamma - \delta n(t)$$

Steady-State Population

Set $\dot{n}(t) = 0$:

$$\bar{n} = \frac{\gamma}{\delta}$$

Population converges to \bar{n} regardless of initial conditions.



Conclusion

- Chapter 9 introduced:
 - Key **continuous-time** analytical tools: notation, growth rates, Maximum Principle.
 - Continuous-time **growth models** and **Poisson process** basics.
 - Computational methods: interpolation, root-finding, optimization, discretizing AR(1).



Thank you!

Questions or comments?

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